48. On Krull's Conjecture Concerning Completely Interally Closed Integrity Domains, III.

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As was kindly called attension by Mr. G. Azumaya, the argument in the previous parts I, II[1] contained a lack, which the writer wants to make up in the following. It was proved, namely, that the Archimedean vector-lattice $\mathfrak{L} = \mathfrak{L}_\Omega$, constructed in I, can not be faithfully represeneted by (finite) real-valued functions, but the representations considered in the proof there were lattice-representations preserving meet and join, and so it was not shown that it can not be order-isomorphically represented by a group of real-valued functions. However, a slight modification of the proof shows this latter too, and thus our counter-example to Krull-Clifford's problem remains valid.

Let A be, as before, a complete Boolean algebra containing a countable set of non-zero and non-atomic elements $v_1, v_2, \ldots, v_i \ldots$ such that for any $a > 0$ in A we have $a \geqq v_i$ with a suitable i; we may take as A, for instance, the complete Boolean algebra of regular open sets of the interval (0, 1). Let $\Omega = \Omega\ (A)$ be its representation space, and let $\mathfrak{L}' = \mathfrak{L}_\Omega$ be the vector-lattice of real- and $\pm\infty$-valued continuous functions on Ω finite except on nowhere-dense sets. Then

Theorem 0. The Archimedean partially ordered (additive) group $\mathfrak{L}$ *can never be order-isomorphically represented by (finite) real-valued functions. In fact, it has no non-trivial order-preserving homomorphic mapping into th ordered additive group of real numbers.*

Proof. Let $g \to \alpha(g)$ $(g \varepsilon \mathfrak{L})$ be an order- and group-homomorphic mapping of $\mathfrak{L}$ into the ordered group of real numbers. Let $\mathfrak{p}$ be an arbitrary point in Ω. We assert that there exists an element g in $\mathfrak{L}$ such that $g \geqq 0$ (or, what is the same, $g(\mathfrak{q}) \geqq 0$ for every $\mathfrak{q} \varepsilon \Omega$), $g(\mathfrak{p}) \geqq 1$ and $\alpha(g) = 0$. Namely, assume the contrary and suppose that $\alpha(g)$ (≠whence) > 0 whenever $g(\mathfrak{p}) \geqq 1$, $g \geqq 0$. Let $w_1 \geqq w_2 \geqq \ldots \geqq w_i \geqq \ldots$ be a monotonic sequence of elements in (the maximal prime dual ideal) $\mathfrak{p}$ such that inf $w_i = 0$ (cf. I, Lemma 1). Then w_1-set $\geqq w_2$-set $\geqq \ldots \geqq w_i$ -set $\geqq \ldots \wedge (w_i$ -set) (no-

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where-dense) $\varepsilon\, \mathfrak{p}$. Let g_i be the characteristic function of the w_i-set. According to our assumption

$$a_i = a(g_i) > 0.$$

Put

$$f(\mathfrak{q}) = 0 \qquad \text{when } \mathfrak{q}\ \varepsilon\ w_1\text{-set},$$
$$f(\mathfrak{q}) = \Sigma_{j=1}^{i}[a_j^{-1}+1] \qquad \text{when } \mathfrak{q}\ \varepsilon\ (w_i\text{-set}) - (w_{i+1}\text{-set}),$$
$$f(\mathfrak{q}) = +\infty \qquad \text{when } \mathfrak{q}\ \varepsilon \wedge (w_i\text{-set}).$$

f lies in our $\mathfrak{L}$. Moreover

$$f \geqq [a_1^{-1}+1]g_1 + [a_2^{-1}+1]g_2 + \dots + [a_i^{-1}+1]g_i$$

for every i, as one readile verifies. Hence necessarily $a(f) = \Sigma_{j=1}^{i}[a_j^{-1}+1]a(g_j) \geqq \Sigma_1^i 1 = i$ for all $i = 1, 2, 3, \dots$, which is of course absurd.

So there exists for each $\mathfrak{p}$ an element $g_\mathfrak{p}$ in $\mathfrak{L}$ such as $g_\mathfrak{p} \geqq 0$, $g_\mathfrak{p}(\mathfrak{p}) \geqq 1$ and $a(g_\mathfrak{p}) = 0$. Let $U_\mathfrak{p}$ be a neighborhood of $\mathfrak{p}$ such that $\mathfrak{q}\ \varepsilon\ U_\mathfrak{p}$ implies $g_\mathfrak{p}(\mathfrak{q}) \geqq 1/2$. As Ω is bicompact it is covered by a finite number of such neighborhoods (of different points), say $U_{\mathfrak{p}1}, U_{\mathfrak{p}2}, \dots, U_{\mathfrak{p}n}$. Put

$$G = g_{\mathfrak{p}1} + g_{\mathfrak{p}2} + \dots + g_{\mathfrak{p}n}.$$

Then $a(G) = 0$ although $G \geqq 1/2$ everywhere in Ω. If now g is an element in $\mathfrak{L}$ everywhere finite, then $-mG \leqq g \leqq mG$ for a suitable natural number m, thus necessarily $a(g) = 0$. Since this is the case for every order-group-homomorphic mapping of $\mathfrak{L}$ into the real number group our first assertion is proved.

In order to obtain the second, further, let g be an element in $\mathfrak{L}$. Consider $g^2 = g(\mathfrak{q})^2$, which also lies in $\mathfrak{L}$. For every natural number i we have $ig \leqq g^2 + i^2$. But $a(i^2) = 0$, by the above, and so $a(g^2) \geqq i\,a(g)$. This implies however $a(g) \leqq 0$. On considering $-g$ wo get $a(g) = 0$.

Remark. The functions in $\mathfrak{L}$ used in the above proof assume $\pm\infty$ and rational integers only. Hence Remark 3 in I also applies to the present formulation.

We have so, even stronger than Theorem 1 in II,

Theorem 1'. The integrity domain R_Ω of polynomial series over $\Omega(A)$, constructed in II, *is completely integrally closed, but has no non-trivial special valuation.*