44. On the Singularities of Analytic Functions with a General Domain of Existence.

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1. Recently S. Kametani and M. Tsuji have investigated the behaviour of a meromorphic function with the set of capacity zero of essential singularities.(1) Let E be a bounded closed set of capacity zero. Suppose that $w=w(z)$ is uniform and meromorphic outside E and has a transcendental singularity at every point of E. Tsuji has found that a theorem of Evans(2) plays an important rôle in such an investigation and obtained systematically some interesting theorems concerning the behaviour of $w=w(z)$. Evans' theorem states that there exists a distribution of positive mass $d\mu(a)$ entirely on E such that

$$(1) \qquad u(z)=\int_E \log\left|\frac{1}{z-a}\right| d\mu(a), \int_E d\mu(a)=1$$

tends to $+\infty$, when z tends to any point of E. Let $v(z)$ be its conjugate harmonic function and put

$$(2) \qquad \zeta=\chi(z)=e^{u(z)+iv(z)}=\rho(z)\,e^{iv(z)}.$$

Let C_r be the niveau curve: $\rho(z)=\text{const.}=r$, then C_r consists of a finite number of simple closed curves surrounding E and moreover there holds

$$(3) \qquad \int_{C_r} dv(z)=\int_{C_r}\frac{\partial u}{\partial n}\,ds=2\pi,$$

where ds is the arc length of C_r and n is the inner normal of C_r. Suggested from Tsuji's proof for the extension of Gross'(3) theorem concerning the principal star-region of an inverse element of $w=w(z)$, the present author

(1) S. Kametani: The exceptional values of functions with the set of capacity zero of essential singularities, Proc. Imp. Acad. **17** (1941), pp. 429–433; On Hausdorff's measure and generalized capacities with some of their applications to the theory of functions, Jap. Journ. Math. **19** (1945), pp. 217–257.

M. Tsuji: On the behaviour of a meromorphic function in the neighbourhood of a closed set of capacity zero, Proc. Imp. Acad. **18** (1942), pp. 213–219; Theory of meromorphic functions in a neighbourhood of a closed set of capacity zero, Jap. Journ. Math. **19** (1944), pp. 139–154.

(2) G. C. Evans: Potentials and positively infinite singularities of harmonic functions, Monatsheft für Math. und Phys. **43** (1936), pp. 419–424.

(3) W. Gross: Über die Singularitäten analytischer Funktionen, Monatshefte für Math. und Phys., **29** (1918), pp. 1–47.

has obtained some results which contribute to such a study, with the aid of the brilliant research of Ahlfors[1] on covering surfaces. Denote by Δ_r the domain exterior to the niveau curve $C_r: \rho(z) = r$ and by F_r its Riemannian image by $w = w(z)$ on the Riemann sphere Σ, with unit diameter, which touches the w-plane at $w = 0$. We denote by $A(r)$ and $L(r)$ the area of F_r and the length of the image of C_r respectively. Let $D_1, D_2, \cdots, D_q$ $(q \geqq 3)$ be q closed circular discs, no two of which have any point in common. Further, we define the defect $\delta(D_j)$ and the ramification index $\theta(D_j)$ of D_j in the same way as Ahlfors and put $\xi = \overline{\lim_{r\to\infty}} \frac{n(r)}{S(r)}$ where $n(r)$ denotes the number of the closed curves of the niveau curve C_r. Then we, first, obtain the following.

Theorem 1. *Let E be a bounded closed set of capacity zero and $w = w(z)$ be uniform and meromorphic outside E. Suppose that $w = w(z)$ has an essential singularity at every point of E. If we define, by the aid of Evans' theorem, the defect $\delta(D_j)$, the ramification index $\theta(D_j)$ and a quantity ξ depending on the number of the closed curves of the niveau curve C_r, then there exists*

$$\sum_{j=1}^{q}\delta(D_j) + \sum_{j=1}^{q}\theta(D_j) \leqq 2 + \xi. \tag{4}$$

Next we shall consider the inverse function $z = z(w)$ of $w = w(z)$ in the neighbourhood of its transcendental singularity Ω. In this case another application of Ahlfors' theory of covering surfaces gives an extension of a theorem of the present author.[2]

Theorem 2. *Let $w = w(z)$ be a uniform meromorphic function outside a bounded closed set of capacity zero of essential singularities and suppose that the inverse function $z = z(w)$ has a transcendental singularity, say Ω, with the projection ω. Denote by Δ_ρ the set of values taken by the branch $z = \varphi_\rho(w)$ of $z = z(w)$ which will define the ρ-neighbourhood Φ_ρ of an accessible boundary point Ω of the Riemann surface Φ associated with $z = z(w)$. We suppose further that Φ_ρ is simply connected. Then, Φ_ρ covers every point infinitely often inside (c) with one possible exception in the same as the case where $w = w(z)$ is a transcendental meromorphic function for $|z| < +\infty$.*

By using theorem 2, it is possible to state a theorem on conformal representation. We have

(1) L. Ahlfors: Zur Theorie der Uberlagerungsflächen, Acta Math., **65**, (1935), pp. 157–194.

(2) K. Noshiro: On the singularities of analytic functions, Jap. Journ. Math. **17** (1940), pp. 37–96. See, esp. theorem 9, p. 95.

Theorem 3. *Let e be a closed set of capacity zero, lying completely inside the unit-circle (c): $|w|<1$ and denote by D the remaining domain obtained by excluding the set e from the disc (c). Let $w=f(z)$ be a mapping function which transforms the unit-circle $|z|<1$ into the universal covering surface $\tilde{D}$ of D in a one-one conformal manner. Further suppose that e contains at least two points. Then the perfect set E of its essential singularities will be of linear measure zero but the capacity of E must be positive.*

Remark. In the case where e consists of a single point, the above theorem does not hold good. To see this fact, we have only to consider a function $w=e^{\frac{z+1}{z-1}}$ which represents $|z|<1$ conformally into the pricked (*punktiert*) circular domain $0<|w|<1$.

2. Let $w=f(z)$ be regular and bounded in the unit-circle $|z|<1$. Fatou's theorem shows that $f(z)$ has a radial limit $\lim_{r\to 1} f(re^{i\theta})=f(e^{i\theta})$ at any point $z=e^{i\theta}$ excluding a possible set of θ-values of measure zero. If the set of $e^{i\theta}$ such that $|f(e^{i\theta})|=1$ be of measure 2π, then after W. Seidel[1] $w=f(z)$ will be called a function of class (U). In the similar way, we can define a function of class (U_ρ), by using the circle $|w|=\rho$ instead of the unit-circle $|w|=1$. We give the following

Lemma *Given a closed set E_z of measure zero on the unit-circle $|z|=1$, we can find a non-constant bounded regular function inside $|z|<1$ which has a boundary value zero at any point belonging to E_z.*

By the above lemma, we have

Theorem 4. *Let $w=f(z)$ be a function of class (U) and $z=\varphi_\rho(w)$ be any branch[1] defined in the circle $|w|<\rho$, ρ being an arbitrary positive number less than unity, of the inverse function $z=\varphi(w)$. Denote by Δ_ρ the set of all values taken by $z=\varphi_\rho(w)$ and map the simply connected domain Δ_ρ upon the unit-circle $|\zeta|<1$ in a one-one conformal manner by a fundamental theorem of conformal representation, then the function*

$$F(\zeta)=f[z(\zeta)] \qquad (5)$$

where $z=z(\zeta)$ denotes the mapping function, will be a function of class (U_ρ); that is, the set of $e^{i\theta}$ such that $|F(e^{i\theta})|=\rho$ must be of measure 2π, $F(e^{i\theta})$ denoting a radial limit of $F(\zeta)$ at $e^{i\theta}$

Applying Theorem 4, we can state

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(1) W. Seidel: On the distribution of values of bounded analytic functions, Trans. of Amer. Math Soc. **36** (1934).

(2) The existence of such a branch is an immediate consequence from Seidel's theorem: if a function $w=f(z)$ of class (U) has at least one singularity z_0 on the unit-circle $|z|=1$, then the cluster set S_{z_0} coincides with the closed circular domain $|w|\leq 1$.

Theorem 5. (An extension of Iversen's theorem[1]). *Let $w = f(z)$ be a function of class (U) and $z = \varphi(w)$ be its inverse function defined in the unit-circle $|w| < 1$. Then for any circle (c): $\left|\frac{w-\omega}{1-\bar{\omega}w}\right| < \rho\,(<1)$ and for any element $P(w; \alpha)$ of $z=\varphi(w)$ whose centre α lies inside (c), we can find a suitable path joining $w=\alpha$ and $w=\omega$ inside (c) along which the element $P(w; \alpha)$ can be continued except possibly at $w=\omega$ (in other words, $z=\varphi(w)$ has Iversen's property within $|w| < 1$).*

As an immediate corollary, we get

Theorem 6. (Seidel)[2] *Let $w = f(z)$ be a function of class (U) and ω be an exceptional value of $w = f(z)$. Then there exists at least one radius along which $w = f(z)$ tends to ω when z tends to its end-point on $|z| = 1$. Moreover, in the case where $w = f(z)$ takes ω only a finite number of times inside the unit-circle, the assertion is also true, provided that $w = f(z)$ has at least one singularity on $|z| = 1$.*

Remark. As an application of Theorem 4, it is possible to give a simple proof for Seidel-Frostman's theorem:[3] Let $w=f(z)$ be a function of class (U) with at least one singularity on the unit-circle $|z| = 1$. Then $w=f(z)$ takes any value, lying inside $|w| < 1$, infinitely often, with a possible exception of w-values of capacity zero, in the interior of the unit-circle $|z| < 1$.

3. Here we will give some extensions of Kametani-Tsuji's theorems. Let E be a bounded closed set of capacity zero, contained entirely inside a domain D and $w=w(z)$ be k-valued algebroidal in the domain D-E such that $w=w(z)$ has at least one transcendental singularity at any point z_0 belonging to E. We can prove the following theorems.

Theorem 7. (An extension of Nevanlinna's theorem).[4] *The cluster set of $w=w(z)$ at $z=z_0$ coincides with the whole w-plane.*

Theorem 8. (An extension of Kametani's theorem).[5] *In any neighbourhood of every point z_0 belonging to E, $w=w(z)$ takes any value infinitely often except a possible set of w-values of capacity zero.*

Theorem 9. (An extension of Tsuji's theorem).[6] *Let Φ_ρ be ρ-neighbourhood of the accessible boundary point Ω of the Riemann covering surface Φ associated with the inverse function $z=z(w)$, supposing that $z=z(w)$ has a transcendental singularity Ω with the projection ω. Then Φ_ρ is a covering sur-*

(1) F. Iversen: Recherches sur les fonctions inverses des fonctions meromorphes, Thèse de Helsingfors, 1914.

(2) W. Seidel: loc. cit.

(3) O. Frostman: Potentiel d'equilibre et capacité des ensembles, Lund (1935), pp. 103–115.

(4) R. Nevanlinna: Eindeutige analytische Funktionen (1936), pp, 106–153.

(5) S. Kametani: loc. cit.

(6) M. Tsuji: loc. cit.

face of the basic surface (c)*:* $|w-\omega|<\rho$ *and covers every point inside* (c) *infinitely often with a possible exception of a set of capacity zero.*

In this case we can also apply Ahlfors' theory of covering surfaces and extend Theorem 2 in the following way:

Theorem 10. *Let* E *be a bounded closed set of capacity zero and* $w=w(z)$ *be k-valued algebroidal outside* E *such that* $w=w(z)$ *has at least one essential singularity at any point* z_0 *belonging to* E*. Suppose that the inverse function* $z=z(w)$ *has a transcendental singularity* Ω *with the projection* ω*. Let* Φ_ρ *be the* ρ*-neighbourhood of the accessible boundary point* Ω *of the Riemann covering surface* Φ *associated with* $z=z(w)$*. Suppose, in addition, that* Φ *is simply connected, then* Φ *covers every point inside* (c)*:* $|w-\omega|<\rho$ *infinitely often except one possible point inside* (c)*.*

The details of the proofs for these theorems will be published in another place.