6. Finite Groups with Faithful Irreducible and Directly Indecomposable Modular Representations.

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The structure of finite groups possessing faithful (isomorphic) irreducible representation (i. r.) in a non-modular field has been determined by K. Shoda[1]; his argument was supplemented by Y. Akizuki. The result is: A finite group $\mathfrak{G}$ possesses a faithful i. r. in an (arbitrary) non-modular field, if and only if for every ideal factor of the product of all minimal abelian invariant subgroups the inequality

$$\text{(S)} \qquad c \leqq m/\lambda$$

is fulfilled, where c, l^m and l^λ denote respectively the number of minimal factors in the ideal factor, the order of the minimal factor, and the number of elements in the $\mathfrak{G}$-automorphism quasifield of the minimal factor.

A somewhat generalized problem to determine those finite groups which have faithful non-modular representations with t irreducible components (i. c.), where t is a natural number, has been considered by M. Tazawa[2]. The result is to replace (S) by

$$\text{(T)} \qquad c-[(t-1)c/t] \leqq m/\lambda$$

Now in the present note we consider modular representations[3]. Here i. r., directly indecomposable representations (d. i. r.) and directly indecomposable components of regular representation (d. i. c. of r. r.) are three classes of representation of particular concern. Our results about faithful representations of these kinds are similar to the above theorems of Shoda and Tazawa, and assume more or less expected forms. Namely:

Theorem 1. Let K be an arbitrary field of charactristic p, and $\mathfrak{M}$ *the product of all abelian minimal subgroups* of order prime to *p in a finite group* $\mathfrak{G}$. *Then:* i) $\mathfrak{G}$ *possesses a faithful d. i. r. (resp. representation with t directly indecomposable components (d. i. c.)) in K if and only if* (S) (*resp.* T)) *is satisfied for every ideal factor in* $\mathfrak{M}$; ii) *The same is also necessary and sufficient in order that* $\mathfrak{G}$ *have a faithful d.i.c. of r. r. (resp. representation decomposed (directly) into t d. i. c. of r. r.) in K;* iii) $\mathfrak{G}$ *has a faithful i. r. (resp. completely reducible representation*

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1) K. Shoda, Über direkt zerlegbare Gruppen, Journ. Fac. Sci. Tokyo Imp. Univ. Section I, Vol. II-3 (1930), § 7; correction, Vol. II-7 (1931).

2) M. Tazawa, Über die isomorphe Darstellung der endlichen Gruppe, Tohoku Math. J. 47 (1940).

3) For modular representations of finite groups, in particular the theory of Brauer-Nesbitt, see the references in M. Osima, 有限群のモヂュラー表現, 日本數學物理學會誌 16 (1942); in the following we shall not however need any deeper part of the theory.

with t i. c.) in K if and only if the same condition for $\mathfrak{M}$ is satisfied and, moreover $\mathfrak{G}$ has no invariant subgroup $\neq 1$ whose order is a power of p.

Perhaps of some interest, though not unexpected, is the corollary that *if $\mathfrak{G}$ possesses a faithful d. i. r. whatsoever then it has a faithful d. i. c. of r. r. (and similarly for representation with t components).* Ot interest is also the relationship between modular and ordinary representations, and in particular the third assertion of the corollary[4] : *If $\mathfrak{G}$ has a faithful i. r. in a modular field then it possesses a same in any non-modular field. If $\mathfrak{G}$ has a faithful i. r. in a non-modular field, then it has a faithful d. i. c. of r. r. in any modular field. If $\mathfrak{G}$ possèsses faithful d. i. r. in two modular fields of distinct characteristics, then it has a faithful i. r. in any non-modular field. (And, exactly the same for representations with t components).*

Also proof runs similarly as in Shoda l. c. (or Tazawa, l. c.); we have only to employ some elementary lemmas on modular representations.

Lemma 1. Let $M(\mathfrak{G})$ be an i. r. (resp. d. i. c. of r. r.) of $\mathfrak{G}$. Then its restriction $M(\mathfrak{N})$ to an invariant subgroup $\mathfrak{N}$ is decomposed directly into a certain number of i. r. (resp. d. i. c. of r. r.) of $\mathfrak{N}$ mutually conjugate in $\mathfrak{G}$. Every i. r. (resp. d. i. c. of r. r.) $m(\mathfrak{N})$ of $\mathfrak{N}$ appears as a direct component in the restriction $M(\mathfrak{N})$ of a suitable i. r. (resp. d. i. c. of r. r.) of $\mathfrak{G}$. Proof: The first assertion concerning i. r. is well known and is in fact valid in a far more general case (where neither $\mathfrak{G}$, $\mathfrak{N}$ nor $(\mathfrak{G}:\mathfrak{N})$ need to be finite)[5]. As to the second (concerning i. r.) we have only to consider, for instance, the representation of $\mathfrak{G}$ induced by $m(\mathfrak{N})$ and its irreducible (not necessarily direct) consituent, whose restriction contains $m(\mathfrak{N})$. Let next $m(\mathfrak{N})$ be a d. i. c. of r. r. of $\mathfrak{N}$, and $N(\mathfrak{G})$ be the representation of $\mathfrak{G}$ induced by it. $N(\mathfrak{G})$ is directly decomposed into a certain number of d. i. c. of r. r. of $\mathfrak{G}$. For, if e is a primitive idempotent in the group algebra $(\mathfrak{N})$ of $\mathfrak{N}$ (over K) such that the left ideal $(\mathfrak{N})e$ defines $m(\mathfrak{N})$, then $N(\mathfrak{G})$ is defined by the direct summand left ideal $(\mathfrak{G})(\mathfrak{N})e=(\mathfrak{G})e$ of the group algebra $(\mathfrak{G})$ of $\mathfrak{G}$. On the other hand the restriction $N(\mathfrak{N})$ is directly decomposed into components conjugate to $m(\mathfrak{N})$ in $\mathfrak{G}$. In fact a representation of a group induced by a representation of its invariant subgroup of finite index is directly decomposed, when restricted to the subgroup, into representations of the subgroup conjugate to the original one. Comparison of these two decompositions shows that each d. i. c. of $N(\mathfrak{G})$, which is at the same time a d. i. c. of r. r., is decomposed directly into in $\mathfrak{G}$ conjugate d. i. c. of r. r. of $\mathfrak{N}$, when restricted to $\mathfrak{N}$. Further the relation $(\mathfrak{G})(\mathfrak{N})=(\mathfrak{G})$ shows that every d. i. c. of r. r. of $\mathfrak{G}$, as well as

4) The first part of the corollary is trivial. In fact we have only to consider an i.r. with the faithful modular constituent. As to the second part a d.i.c. of modular r.r. containing the faithful non-modular i.r. suffices, as our proof below shows. It is rather likely (and desirable) that a direct representation theoretical proof can be given to the third (as well as to the second) assertion of the corollary.

5) Cf. A. H. Clifford, Representations induced in an invariant subgroup, Ann. Math. 38 (1937).

that of $\mathfrak{N}$, appears in such circumstance (with suitable $m(\mathfrak{N})$). Now the assertions about d. i. c. of r. r. in the lemma are ready to verify.

Lemma 2. Let $\mathfrak{G}$ and $\mathfrak{N}$ be as in Lemma 1. Let $M(\mathfrak{G})$ be a representation of $\mathfrak{G}$ such that its restriction to $\mathfrak{N}$ is directly decomposed into a certain number of mutually in $\mathfrak{G}$ conjugate representations $m_1(\mathfrak{N})$, $m_2(\mathfrak{N})$, ..., $m_n(\mathfrak{N})$. Let $\mathfrak{C}$ be the kernel of the representation $M(\mathfrak{G})$, and $\mathfrak{D}_j$ the kernel of $m_j(\mathfrak{N})$. Then $\mathfrak{D}_1 \frown \mathfrak{D}_2 \frown \cdots \frown \mathfrak{D}_n = \mathfrak{N} \frown \mathfrak{C}$, and this is the greatest invariant subgroup of $\mathfrak{G}$ contained in one of $\mathfrak{D}$'s, say $\mathfrak{D}_1$. So, if $M(\mathfrak{G})$ is faithful, then $\mathfrak{D}_1$ contains no invariant subgroup $\neq 1$ of $\mathfrak{G}$. Conversely, if $\mathfrak{D}_1$ contains no invariant subgroup $\neq 1$ of $\mathfrak{G}$ then the restricted representation $M(\mathfrak{N})$ is faithful (on $\mathfrak{N}$).

Let now $\mathfrak{N}$ be in particular the invariant subgroup generated by the totality of minimal invariant subgroups of $\mathfrak{G}$. Then a representation of $\mathfrak{G}$ is faithful already when it is so on $\mathfrak{N}$. Thus we see: $M(\mathfrak{G})$ is faithful if and only if the kernel $\mathfrak{D}_1$ of $m_1(\mathfrak{N})$ contains no invariant subgroup $\neq 1$ of $\mathfrak{G}$.

Let next $\mathfrak{N}$ be an invariant subgroup generated by (not necessarily all but) some minimal invariant subgroups in $\mathfrak{G}$. It is uniquely decomposed into the direct product

$$\mathfrak{N} = \mathfrak{K} \times \mathfrak{L}_1 \times \mathfrak{L}_2 \times \cdots\cdots \times \mathfrak{L}_h$$

of a product $\mathfrak{K}$ of non-abelian simple invariant subgroups and abelian invariant subgroups $\mathfrak{L}_k$ of prime power orders with different primes l_k. Each $\mathfrak{L}$ is of type $(l, l, \ldots\ldots l)$. If $l \neq p$ its representation is completely reducible and its cyclic factor groups, and only those, have faithful i. r. But if $l = p$ its unique i. r. (resp. d. i. c. of r. r.) is unit representation (resp. faithful). On the other hand each simple factor in $\mathfrak{K}$ certainly has a faithful d. i. c. of r. r., while it possesses a faithful i. r. if and only if its order is not a power of the characteristic p. Now, every invariant subgroup $\mathfrak{D}$ of $\mathfrak{N}$ is regular with respect to the above decomposition of $\mathfrak{N}$, that is, $\mathfrak{D}$ is decomposed into subgroups $\mathfrak{S}$, $\mathfrak{T}_k$ of $\mathfrak{K}$, $\mathfrak{L}_k$:

$$\mathfrak{D} = \mathfrak{S} \times \mathfrak{T}_1 \times \mathfrak{T}_2 \times \cdots\cdots \times \mathfrak{T}_h.$$

The Kronecker product of representations of $(\mathfrak{L}/\mathfrak{T})$'s and simple factors of $\mathfrak{K}$ is faithful for $\mathfrak{N}/\mathfrak{D}$ if and only if each of them is faithful (on each subgroup). Moreover, in case the underlying field K is algebraically closed, the Kronecker product of i. r. of direct factors is an i. r. of the product, and every i. r. of the (direct) product is obtained in this way. Exactly the same holds also for d. i. c. of r. r. However, a (finite) group has a faithful i. r. (resp. d. i. c. of r. r.) in a field if and only if it has a such in its algebraic closure[6]. These together enable us to combine the above results about direct factors into

Lemma 3. The invariant subgroup $\mathfrak{N}$ generated by some minimal invariant subgroups of $\mathfrak{G}$ has an i. r. (resp. completely reducible representation with t i. c.) in K whose kernel contains no invoriant subgroup $\neq 1$ of $\mathfrak{G}$, if and only if 1) every $\mathfrak{L}$ with $l \neq p$ possesses an

6) Observe, in case K is imperfect, that the group algebra is separable.

invariant subgroup (resp. t invariant subgroups, not necessarily distinct) with cyclic factor group (resp. cyclic factor groups) which (resp. whose intersection) contains no invariant subgroup$\neq 1$ of $\mathfrak{G}$, 2) no l (with $\mathfrak{L}\neq 1$) coincides with p, and 3) no simple component of $\mathfrak{K}$ has a power of p as its order. (Here 2) and 3) may be combined into one condition: 0) $\mathfrak{N}$ has no invariant subgroup$\neq 1$ whose order is a power of p.) $\mathfrak{N}$ possesses a faithful d. i. c. of r. r. (resp. representation decomposed directly into t d. i. c. of r. r.) in K if and only if the condition 1) is fulfilled.

Applying this lemma again to the product $\mathfrak{N}$ of all minimal invatiant subgroups we have now

Theorem 1'. Let $\mathfrak{M}$ be as in Theorem 1 and decompose it into

$$\mathfrak{M}=\mathfrak{L}_1\times\mathfrak{L}_2\times\ldots\ldots\times\mathfrak{L}_g$$

where $\mathfrak{L}_i$ are subgroups of prime power orders with different primes l_i $(\neq p)$. $\mathfrak{G}$ possesses a faithful i. r. (resp. completely reducible representation with t i. c.) in K, if and only if $\mathfrak{G}$ has no invariant subgroup$\neq 1$ whose order is a power of p and moreover the condition 1) (of Lemma 3) is satisfied. $\mathfrak{G}$ has a faithful d. i. c. of r. r. (resp. representation directly decomposed into t d. i. c. of r. r.) in K, if and only if 1) is fulfilled, and exactly the same for a faithful d. i. r. general (resp. representation with t d. i. c.).

Only the part concerning general d. i. r. is left to be shown. Let $M(\mathfrak{G})$ be a such of $\mathfrak{G}$. Its restriction $M(\mathfrak{M})$ to $\mathfrak{M}$ is completely reducible, since the order of $\mathfrak{M}$ is prime to p, and the irreducible components of $M(\mathfrak{M})$ are all conjugate in $\mathfrak{G}$. Take namely the primitive idempotent e belonging to one of them in the commutrtive group algebra $(\mathfrak{M})$, and let $e, e', e'',\ldots\ldots$be its distinct conjugates; they are orthogonal. Denote their sum by E. E is an idempotent and $\mathfrak{m}=E\mathfrak{m}+(1-E)\mathfrak{m}$ gives a direct decomposition of the representation module $\mathfrak{m}$ of $M(\mathfrak{G})$. Hence necessarily $\mathfrak{m}=E\mathfrak{m}$, and this gives our assertion. Thus Lemma 2 is also applicable to $M(\mathfrak{G})$ with respect to $\mathfrak{M}=\mathfrak{N}$. The necessity of the condition 1) is now immediate. But the sufficiency is contained in the more precise assertion concerning d. i. c. of r. r.

The conditions in Theorem 1' are all of purely group theoretical nature, and in that sense Theorem 1' already gives a solution to our problem; also two corollaries alluded above follow already from Theorem 1'. But we may replace 1) by (S) (resp. (T)), to obtain Theorem 1, in virtue of the (purely group theoretical) lemma (of Shoda-Akizuki): Let $\mathfrak{A}$ be a (Γ-) ideally completely reducible abelian group of type $(l, l,\ldots\ldots l)$ (l a prime) with an operator group Γ: $\mathfrak{A}=\mathfrak{A}_1\times\mathfrak{A}_2\times\ldots\ldots\times\mathfrak{A}_g$ ($\mathfrak{A}_r$ Γ-simple and all mutually Γ-isomorphic). The condition (S) is necessary and sufficient for the existence of a subgroup with cyclic factor group and containing no Γ-subgroup$\neq 1$; (together with a combinatorial lemma of Tazawa, in case $t>1$, that the largest number s, such that we can divide c symbols into t parts (not necessarily without common symbols) with s symbols in each, so as no symbol is common to all, is $[(t-1)c/t]$).