17. Fundamental Theory of Toothed Gearing (II).

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Assume that a curve K is oriented to a certain direction. Take any two points P_0 and P on K. We say that the arc length P_0P is positive or negative according as P exists at the positive side or negative side of P_0. The orientation of the tangent to K at any point may be defined in usual manner in accordance with that of K. Let C be an arbitrary point. We give the length of the segment PC of the straight line connecting P with C a positive or negative sign according as C exists on the left side or right side of the tangent to K at P. Referring to a pair of pitch curves K_1 and K_2 we shall assume that they are oriented in the same sense, that is, the common tangent at every instant has the same sense even if observed as a tangent of K_1 or of K_2.

§ 1. Analytical representation of profile curves.

Given an oriented curve K and determined on it an arbitrary point P_0 as origin, then we can indicate the position of any point P on K by the arc length P_0P which we shall denote by s and call the abscissa of P on K. Now consider a family of circles with centers on K. This family is established when the following relation is given:

$$r=f(s) \tag{1}$$

between the abscissa s of any point P on K and the radius r of the circle with P as the center. If this family possesses envelopes, we can determine one F of them, when the sign of r in Equation (1) is indicated. In this case $|f(s)|$ is a one-valued continuous function of s and we may assume further it is differentiable as regards s such that $|f(s)|\neq 0$. Next, we denote by θ the angle between the perpendicular drawn from an arbitrary point P on K to the curve F and the oriented tangent to K at P. We shall also give θ the same sign as that of r. Then we have the following simple relation among these three quantities r, s and θ:

$$\frac{d|r|}{ds}=-\cos\theta\,,\quad sgn(\theta)=sgn(r)\,. \tag{2}$$

Now we shall conclude the proof of Theorem 2 in the report (I). The pair of envelopes F_1 and F_2 which we already determined in the first half of the proof may be represented by the same equation (1). Hence the direction of two perpendiculars drawn from any common pitch point to F_1 and F_2 evidently coincide by the relation (2). Thus the theorem is established.

By this discussion we can represent each of an arbitrarily given pair of profile curves by the same equation (1). The absolute-valued function $|f(s)|$ is one-valued and continuous as regards s. We assume, after this, that $|f(s)|$ is differentiable at least twice as regards s such that $|f(s)| \gtrless 0$.

The equation of any profile curve F^* parallel to a given profile curve F with Equation (1) is given by

$$(3) \qquad r^* = f(s) + a ,$$

where a represents an arbitrary constant.

Now we may understand from another point of view Equation (1) of the given profile curve F as the expression giving the length r of the segment PC of the straight line connecting any point P on K_r with C, where K_r means the rolling curve and C the drawing point both of which are determined for F by Theorem 3 in the report (I). Let $a_r = a_r(s)$ be the natural equation of K_r and θ be the angle between the straight line PC and the tangent to K_r at P. Then we have the following relations:

$$(2) \qquad \frac{d|r|}{ds} = -\cos\theta , \quad sgn(\theta) = sgn(r)$$

and

$$(4) \qquad \frac{d\theta}{ds} = \frac{\sin\theta}{|r|} - \frac{1}{a_r} .$$

From (4) follows

$$(5) \qquad \frac{1}{a_r} = \frac{\sin\theta}{|r|} - \frac{d\theta}{ds} ,$$

and from (2) we have

$$(6) \qquad \theta = \cos^{-1}\{-|f(s)|'\}, \quad sgn(\theta) = sgn(f(s))$$

and then

$$(7) \quad \sin\theta = sgn(f(s))\sqrt{1-\{|f(s)|'\}^2}, \quad \frac{d\theta}{ds} = sgn(f(s))\frac{|f(s)|''}{\sqrt{1-\{|f(s)|'\}^2}} .$$

Substituting (7) into (4) we can derive the natural equation of K_r in the following form:

$$(8)\qquad a_r=a_r(s)=sgn(f(s))\cdot\frac{|f(s)|\sqrt{1-\{|f(s)|'\}^2}}{1-\{|f(s)|'\}^2-|f(s)|\cdot|f(s)|''}$$

$$=\frac{f(s)\sqrt{1-\{f'(s)\}^2}}{1-\{f'(s)\}^2-f(s)\cdot f''(s)}.$$

Conversely, if the natural equation $a_r=a_r(s)$ is given at first, then we derive Equation (1) of the profile curve F corresponding to K_r by solving differential equation (8) for $f(s)$.

§ 2. Necessary and sufficient condition for profile curves (3).

Given Equation (1) for a profile curve F, then in consequence of (2) we have a necessary condition of the form (9) for $f(s)$:

$$(9)\qquad -1\leqq|f'(s)|\leqq 1.$$

Conversely, we can prove that the above condition (9) is sufficient in order that Equation (1) repsesents a profile curve. For, if (9) be given, we can define a one-valued continuous (and differentiable) function θ of s using (2), namely, (6). On the other hand we can determine a curve K_r by giving (8) as its natural equation. Transforming (8) we have (5), accordingly (4). By both (2) and (4) one point C is determined to K_r. When we roll K_r along the pitch curve K, we have a roulette F of the point C as a profile curve with Equation (1). Thus we have:

Theorem 1. *In order that a profile curve be given by the equation $r=f(s)$, where $|f(s)|$ is one-valusd, continuous and differentiable twice with regard to s, the arc length of a pitch curve, such that $|f(s)|\gtrless 0$, it is necessary and sufficient that the inequality $|f'(s)|\leqq 1$ holds in the given interval of s.*

We shall notice that the method used for the proof of sufficiency of Theorem I can be applied for the analytical proof of the fact that the condition (e) (see the report (I)) is derived from the condition (b), accordingly (d), and we have already discussed the fact geometrically.

Expressing Theorem 1 in other words we have

Theorem 2. *Given a family of circles with centers on a curve K whose radii are given by a function $r=f(s)$ one-valued and differentiable twice with regard to s, the length of the arc of K. In order that the family possesses an envelope. it is necessary and sufficient that the inequality $|f'(s)|\leqq 1$ holds in the given interval of s.*

§ 3. Paths of contact.

Consider an arbitrary straight line T_0 and a point P_0 on it. Let us denote by Γ the locus of the point of contact of the profile curves F_1 and F_2 when we roll the pitch curves K_1 and K_2 one along the other keeping them to be always touching T_0 at P_0. Γ is called the path of contact. As the profile curve is continuous, its path of contact is of course continuous. We can represent Γ by a polar equation

$$(10) \qquad r=g(\theta), \quad sgn(r)=sgn(\theta)$$

using P_0 as a pole and T_0 as an initial line. The radius vector r in this case has a sign, and the angle θ belongs to some interval contained from $-\pi$ to π. The function $|g(\theta)|$ is continuous in this interval.

The path of contact Γ^* of the profile curve F^* which has the distance a from the given profile curve F and is parallel to it is given by

$$(11) \qquad r^*=g(\theta)+a$$

using the function $g(\theta)$ of Equation (10) which defines the path of contact Γ of F. But in this case $sgn(r^*)=sgn(\theta)$ does not hold necessarily. Comparing (11) with (10) we have:

The paths of contact of two parallel profile curves are concoid curves of each other.

Now if we give the equation of a profile curve F by (1), Equation (10) of the path of contact Γ is derived from (1) and (2) by eliminating s. Conversely, if Equation (10) of Γ is given, we obtain the equation of F eliminating θ from (2) and (10).

From now on we shall assume, without loss of generality, that $f(s)$ is a continuous function with a definite sign and accordingly $g(\theta)$ is so. The function $g(\theta)$ is not necessarily one-valued as regards θ, although $f(s)$ is so as regards s, However, when s, a function of θ, obtained by solving (2) is one-valued, continuous and differentiable, we obtain a one-valued, continuous and differentiable function $g(\theta)$ by substituting s into the function $f(s)$, because $f(s)$ is one-valued, continuous and differentiable (twice) as regards s. Conversely, if $g(\theta)$ is one-valued, continuous and differentiable, then there holds the following relation of s and θ:

$$(12) \qquad \frac{ds}{d\theta}=-sgn(\theta)\frac{g'(\theta)}{\cos\theta}.$$

Accordingly s, the function of θ given by integration of the right side of (12) as regards θ, is one-valued, continuous and differentiable, that is, θ is a monotone function of s and consequently s of θ. Hence we have:

Theorem 3. *A necessary and sufficient condition that the function $g(\theta)$ which defines a path of contact and has a definite sign is one-valued, continuous and differentiable is that θ is a continuous monotone function of s and consequently s of θ.*

Now suppose $g(\theta)$ is a function which is one-valued, continuous and differentiable, then we have its derivative $g'(\theta)$ from (12) as follows:

$$g'(\theta) = -sgn(\theta)\cos\theta\frac{ds}{d\theta}. \tag{13}$$

We shall divide the range of θ into two parts, namely, one part within $0<|\theta|<\frac{\pi}{2}$ and the other part within $\frac{\pi}{2}<|\theta|<\pi$. In the respective intervals, $\cos\theta$ is always positive or negative, accordingly by (13) the sign of $g'(\theta)$ coincides with that of $-sgn(\theta)\frac{ds}{d\theta}$ or opposite to. Furthermore by Theorem 3 s is a continuous monotone function in the whole range of θ, so the sign of $-\frac{ds}{d\theta}$ is definite. Hence we have the following fact:

If the function $g(\theta)$ which defines a path of contact and has a definite sign is one-valued, continuous and differentiable, then it is necessarily a continuous monotone function in the respective intervals belonging to the quadrant $0<|\theta|<\frac{\pi}{2}$ or $\frac{\pi}{2}<|\theta'<\pi$. And conversely.

§ 4. Necessary and sufficient conditions for paths of contact.

When the function $g(\theta)$ defining the path of contact is not one-valued, we shall divide the range of θ into several intervals and may consider $g(\theta)$ is one-valued in the respective intervals. Tberefore we can from the beginning deal with a one-valued and continuous function $g(\theta)$ without loss of generality.

Now suppose a pair of pitch curves are given and besides a curve Γ—$r=g(\theta)$— is taken. We shall discuss whether a pair of profile curves having Γ as its path of contact may exist or not, at this time, however, we assume that $g(\theta)$ is a one valued, continuous and differentiable function with a definite sign. This problem is equivalent to the determination of the condition that such a quantity may be determined as it satisfies the relation (2) for the pair of quantities r and θ given by Equation (10), and that moreover at this time the function obtained by eliminating θ from (2) and (10) may become a one-valued continuous function of s.

If, indeed, Γ is the path of contact, then as we have explained in § 3, Equation (12) holds and s is a continuous, differentiable and monotone function of θ (and consequently θ of s).

Conversely, when the right side of Equation (12) is integrable, then from it we can derive a function s which is one-valued and differentiable as regards θ, and denote it by $s=s(\theta)$. Next, in order that one-valued and continuous function $r=f(s)$ can be derived from one-valued continuous functions $r=g(\theta)$ and $s=s(\theta)$ by eliminating θ it is sufficient that the inverse function of $s(\theta)$, say $\theta=\theta(s)$, is one-valued and continuous, that is, s is a continuous monotone fnnction of θ (and consequently θ of s). Thus we have the following:

Theorem 4. *In order that a path of contact for a pair of profile curves be given by a function $r=g(\theta)$ which is one-valued, continuous and differentiable and has a definite sign, it is necessary and sufficient that $g(\theta)$ be a continuous monotone function in the respective partial intervals belonging to the quadrant $0<|\theta|<\frac{\pi}{2}$ or $\frac{\pi}{2}<|\theta|<\pi$, and the function $\frac{g'(\theta)}{\cos\theta}$ is integrable in the whole range.*

Now we may understand from another point of view that Equation (10) of the path of contact Γ is the expression giving the relation between r and θ, in which r is the length of the segment of the straight line connecting any point P on the rolling curve K_r determined to the profile curve F with the drawing point C, and θ is the angle between the straight line PC and the tangent to K_r at P. Let a_r be the radius of curvature of K_r at P, then from (5) and (2) it follows

$$\frac{1}{a_r}=\frac{\sin\theta}{|r|}+\frac{\cos\theta}{\frac{d|r|}{d\theta}}. \tag{14}$$

On the other hand, however, the quantity a_r given by (14) is the length measured from the pole P_0 to the point M along the straight line N_0 drawn passing through P_0 perpendicularly to the initial line T_0, where M is the point of intersection of N_0 and the normal CM drawn to the curve Γ at any point $C(r, \theta)$ on Γ. Hence we have

Theorem 3. *Let N_0 be the perpendicular drawn to the initial line T_0 at the pole P_0. The length of the segment of N_0 between P_0 and the point M at which the normal to the path of contact Γ at any point C on Γ intersects with N_0 is equal to the radius of curvature of the rolling curve K_r at the pitch point corresponding to C.*

Move the rolling curve K_r keeping it to be always touching the straight line

T_0 at the point P_0. It would be easily understood that the locus of the drawing point C fixed at K_r is the path of contact. In this case the evolute of K_r, denoted by N_r, makes a rolling contact motion by Theorem 5 along the straight line N_0 drawn perpendicularly to T_0 passing through P_0, in other words: the roulette Γ drawn by the point C at the rolling motion of the curve N_r along N_0 is the very path of contact. Consequently we have the following characterization of a profile curve and its path of contact:

Any profite curve and its path of contact characterized as the roulettes of the same one point which is fixed at a suitably taken curve K_r and its evolute N_r, when K_r and N_r roll without sliding along the pitch curve K and an arbitrarily determined normal of K respectively.

Let $a_r = a_r(s)$ be the natural equation of the curve K_r and assume the function $a_r(s)$ is differentiable, then the natural equation $a_r^* = a_r(s^*)$ of N_r is given in the form:

$$s^* = a_r(s), \quad a_r^* = a_r(s)a_r'(s).$$

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