30. *On Finite Groups, Whose Sylow Groups Are All Cyclic.*

By Kinya HONDA.

(Comm. by Z. SUETUNA, M.J.A., May 12, 1949.)

In this paper we shall call those finite groups whose Sylow groups are all cyclic, *hypercyclic*. The group of order may also be called hypercyclic. The aim of this paper is to study the structure of these groups and then to find out all hypercyclic groups of a given order. The proofs of the results will be published afterwards elsewhere.

§ 1. Structure of Hypercyclic Groups.

When a group G has a subgroup A and a normal subgroup B, and $AB=G$ $AB=E$, where E is the unit subgroup of G, then we shall say that G *has a splitting into A and B*, and shall denote it by

$$G=(A, B\sigma_0).$$

Every element a of A gives rise to an automorphism τ $(xaxa, x \in B)$ of B. Let τ_a correspond to a, then we have a homomorphic mapping in the group of automorphisms of B from the group A. The σ in (1) should be understood as a symbol of this marpping. The elements of A, permutable with all elements of B, form a normal subgroup F of G, and will be called the *foundation group of the splitting* (1) The index $[A:F]$, if it is fnite will be called the *index of the splitting* (1).

Henceforth we shall study only finite groups. When afnite group G has a splitting into G_0 and P, where P is a cyclic group and at the same time a Sylow group of G, then we shall call it a *simple splitting* of G. Let F be the foundation group then G/F is cyclic. Now we have following theorems ;

Let

$$G=(G_0, P/\sigma)$$

be a simple splitting of G, and F be the foundation group.

1) **First Reduction Theorem on Subgroups.** *Any subgroup H can be expressible in a form*

$$H=a^{-1} HP' a,$$

mhere H and P are subgroups of G_0 and P respectively, and a is an element of P.

2) **Second Reduction Theorem on Subgroups.** *If*

(2) $$a'H'P'a' \geq a''^{-1}H''P''a''$$

where a' and a'' are elements of P, and H', H'' are subgroups of G_0, and P', P'' are subgroups of P, then

$$H_0' \geq H_0'', \quad P' \geq P'',$$

and in case of H'' F

$$a' \ a''^{-1} \varepsilon \ F',$$

and conversely from these three conditions we obtain (2).

3) *If N is a normal subgroup of G, then N has a splitting*

$$N=(N_0, \ P'/\sigma),$$

where $N_0=N \wedge G$, $P'=N \wedge P$, and in case of $P' \neq P$

$$N_0 \leq F.$$

Conversely, let N_0 and P' be normal subgroups of G_0 and P respectively. Then N_0 P' will be a normal subgroup of G, when either $P'=P$ or $N_0 \leq F$.

4) *Let H_0 and P' be subgroups of G_0 and P respectively. Then, in cose of $H_0 \leq F$*

$$N_G(H_0P')=N_{G_0}(H_0) \cdot P,$$

and in another case

$$N_G(H_0P')=N_{G_0}(H_0) \cdot P,$$

5) **Reduction Theorem on Commutator Groups.** *Let C and C_0 be the cnmmutator groups of G and G_0 repectively. Then, in case of $G=G_0 \times P$,*

$$C=C_0$$

and in another case

$$C=C_0 \times P.$$

Now we shall study the structure of hypercyclic groups. And subgroup of a hypercyclic group is hypercdclic, and any group that is homomorphic to a hypercyclic one, is also hypercyclic. Let G be hypercyclic H be a subgroup of G, and N be a normal subgroup of G, then $[H \mid N]=[H] \vee [N]^{(2)}$ and $[H \mid \mathrm{N}]=[H] \wedge [N]^{(2)}$. Consequently, if G has a normal subgroup of order m, there is no more subgroup of order m other than N. If N' is a normal subgroup of N, which is a normal subgroup of G, then N' is a normal one of G.

When there exist two systems of subgroups of a group G

$$A_1, \ A_2, \cdots\cdots, A_{r-1}, \ A_r$$

$$G_2, G_3, \cdots\cdots, G_{r-1}, G_r \parallel G,$$

and

$$G_2=(A_1, A_2/\sigma_1), G_3=(G_2, A_3/\sigma_2), \cdots\cdots, G_r=(G_{r-1}, A_r/\sigma_{r-1}),$$

then we say that G *has a successive splitting into* $A_1, A_2, \cdots\cdots A_r$, and shall denote it by

$$(3) \qquad G=(A_1, A_2, \cdots\cdots, A_r/\sigma_1, \sigma_2, \cdots\cdots, \sigma_{r-1}).$$

Let the index of the splitting $G_{i+1}=(G_i, A_{i+1}/\sigma_i)$ be f_i $(i=1,2,\cdots\cdots, r-1)$, then we shall call the system of numbers $f_1, f_2, \cdots\cdots, f_{r-1}$ the *system of indices of the splitting* (3)

The systematic study on hypercyclic groups is based upon the following

First Splitting Theorem. *Let the factorization into primepowers of the order n of a hypercyclic group G be* $n=P_1^{e1}\ P_2^{e2}\cdots\cdots P_r^{er}$, *where* $p_j\ p_i-1$ $(1\leqq i<j\leqq r)$. *Then there exists a successive splitting*

$$(4) \qquad G=(P_1, P_2, \cdots\cdots, P_r/\sigma_1, \sigma_2, \cdots\cdots, \sigma_{r-1}),$$

where every P_i *is a (properly taken) p-Sylow group of* G.[4]

If $p_1<p_2<\cdots\cdots<p_r$, of course $p_j\ p_i-1$ $(1\leqq i<j\leqq r)$. In this particular case the splitting (4) will be called the *fundawental splitting of* G.

From this theorem and the reduction theorem on commutator groups we can easily obtain the

Second Splitting Theorem. *Every hypercylic group G has a splitting*

$$G=(H, N/\sigma),$$

where H, N are cyclic, and $[H]\wedge[N]=1$. *One can take the commutator grnup of G as this group N. Couversely such a group is, of coures, hypercyclic.*

This theorem is equivalent to the theorem of H. Zassenhaus.[5]

From the first reduction theorem on subgroups and the first splitting theorem we can obtain the somewhat interesting

Conjugateness Theorem. *Let* H_1 *and* H_2 *be sudgroups of a hypecrcycic group G, and* $[H_1]$ *be a multiple of* $[H_2]$ *Then there exists an element x in G such that* $x^{-1}H_1x\geq H_2$ *Consequently two sudgroups of the same order are conjugate in G.*

§ 2. **L-Similar Classes of Hypercyclic Groups.**

We will attempt to solve the problem of finding out all hypercyclic groups of a given order.

The lattice. which is formed by all subgroups of a group G, will be denoted by $L(G)$. Two finite groups G and G of the same order will be called *L-similar,* and will be denoted by $G\sim G$, when there exists a lattice isomorphic correspondence σ of $L(G)$ and $L(G)$ with the following properties :

1) If $H\longleftrightarrow H$ $(H\leq G,\ H\leq G)$ by ρ, then $[H]=[H]$.

2) If $H\longleftrightarrow H$ $(H\leq G,\ H\leq G)$ by ρ, then $N_G(H)\longleftrightarrow N_G(H)$.

Two L-silimilar groups have almost same structures, but are not necessarily isomorphic.

L-Similarity Theorem. *Let G and G be hypercyclic groups of the same order, and G have a successive splitting*

$$G=(P_1,P_2,\cdots\cdots,P_r/\sigma_1,\sigma_2,\cdots\cdots,\sigma_{r-1}),$$

where P_i is a p-Sylow group of G $(i=1,2,\cdots\cdots,r)$, and the system of indices of this splttting be f_1'$f_2,\cdots\cdots,f_{r-1}$. Then $G\sim G$, if and only if G has a successive splitting of a form

$$G=(P_1,P_2,\cdots\cdots,P_r/\sigma_1,\sigma_2,\cdots\cdots,\sigma_{r-1}),$$

where P_i is a p_i-Sylow group of $G(i=1,2,\cdots\cdots,r)$, and the system of indices of this splitting is $f_1,f_2,\cdots\cdots,\ f_{r-1}$.

Corollary. *Two hypercyclic groups of the same order are L-similar, if and only if the systems of indices of their fundumental aplittings coincide with each other.*

We can now find out all L-similar classes of hypercyclic groups of a given order.

Let n be a given integer, and the factorization into prime-powers of n be $n=p_1^{e1}p_2^{e2}p_r^{er}$, where $p_j\ p_i-1$ $(1\leqq i<j\leqq r)$· Then, by the first splitting theorem any hypercyclic group of order n has a splitting

$$G=(P_1,P_2,\cdots\cdots,P_r/\sigma_1,\sigma_2,\cdots\cdots,\sigma_{r-1}),$$

where P is a p-Sylow group. Let the system of indices of this splitting be $f_1,f_2,\cdots\cdots,f_{r-1}$. If we fix the permutation $p_1,p_2,\cdots\cdots,p_r$, then the system $f_1,f_2,\cdots\cdots,f_{r-1}$ is uniquely determined by the group G. Therefore, by the L-similarity theorem, we can express the L-similar class, in which G belongs, by a symbol $(f_1,f_2,\cdots\cdots,f_{r-1})$. Let $g_1=p_1^{e1}\wedge(p_2-1),g_2=p_1^{e1}p_2^{e2}\wedge(p_3-1),\cdots\cdots,g_{r-1}=p_1^{e1}p_2^{e2}\cdots\cdots p_{r-1}^{er-1}\wedge(p_r-1)$, then of course

1) $f_i\mid g_j\cdot$ $(i=1,2,\cdots\cdots,r-1)$

2) If $f_k\neq 1$, then $p_{k+1}\ f_j$. $(j=1,2,\cdots\cdots,r-1)$

The latter can be obtained from the the reduction theorem on commutator groups. Conversely, for any system of $r-1$ integers $f_1,f_2,\cdots\cdots,f_{r-1}$, which fulfils the conditions 1) and 2), there exists a L-similar class of hypercyclic groups of order n, that is expressible by the symbol $(f_1,f_2,\cdots\cdots,f_{r-1})$. Therefore, the problem of finding out all L-similar classes has, in a sense, been solved.

Finally, we will find out all (not isomorphic to each other) hypercyclic groups in a given L-similar class.

Let $G\sim G$, and a p-Sylow group P of G be a normal subgroup of G, then G has a simple $G=(G_0, P/\sigma)$ and G has a simple splitting $G_0=(G_0, P/\sigma)$, where P is the p-Sylow normal subgroup of G and $G_0\sim G_0$, $P\cong P$. If, in particular, $G\cong G$, then $G_0\cong G_0$.

Now we take (fixed) hypercyclic group G_0 and a (fixed) cyclic p-group P, where $[G_0]\wedge[P]=1$. Let the commutator group of G_0 be C_0, then G_0/C_0 is cyclic. Let the order of P be p^e ($e\geqq 1$), and we take a subgroup F of G_0 such as $F+C_0$ $[G_0: F]|p-1$. In the group of automorphisms of P a subgroup of order $[G_0: F]$ is uniquely determined, and is a cyclic one. Let it be generated by τ and we fix this automorphism τ. Now, if, for an element d of G_0, dF is a generating element of G_0/F, then d can determine a hypercyclic group $(G_0, P/d, \tau)$ followingly :

1) $(G_0, P/d, \tau)$ is a set of $(a\ \ b)$. $(b \in P)$

2) The product of any two elements (a_1, b_1) and (a_2, b_2) is defined by

$$(a_1, b_1)(d^i a_2, b_2)=(a_1 d^i a_2, b_1^{\tau^i} b_2). \ (a_2 \in F)$$

We shall call the centraliser of the commutator group of a (hypercyclic) group G the *ramification kernel* of G.

Ramification Theorem. *We have*

$$(G_0, P/d_1, \tau)\cong(G_0, p/d_2, \tau),$$

if and only if

$$d_1\ d_2^{-1} \in R_0\ F,$$

where R_0 is the ramification kernel of G_0.

Therefore, the problem of finding out all (not isomorphic to eachother) hypercyclic groups of a given order has in a sense, been solved.

On ramification kernels we have following

Reduction Theorem on Ramification Kernels. *Let $G=(G_0,\ P/\sigma)$ be a simple splitting of a finite group G, and F be its foundation group, and R, R_0 be ramification kernels of G, G_0 respectively. Then we have*

$$R=(R_0-F)\times P.$$

From this theorem we can easily obtain the

Enumeration Theorem. *The number of all (not isomorphic to each other) hypercylic groups in am L-similar class $(f_1, f_2, \cdots\cdots, f_{r-1})$*

is (for $r \geqq 3$)

$$\prod_{i=1}^{r-2} \varphi(f_1 \vee f_2 \vee \cdots\cdots \vee f_i) \wedge f_{i+1}),$$

where φ is Fuler's function and it is when $r=1$ or 2.

---

(1). $N_G(H)$ means the normaliser of H in G.

(2). $[H]$ means the order of H, and $[H] \vee [N]$ means the l.c.m. of $[H]$ and $[N]$.

(3). $[H] \wedge [N]$ means the g.c.f. of $[H]$ and $[N]$.

(4). W. Burnside : Theory of Groups. 1 edit. (1897) p. 352.

(5). H. Zassenhaus : Über endliche Fastkörper. Abh. Semin. Hamburg, 11 (1936) p. 19
8. and the same author's Lehrbuch der Gruppentheorie. Bd. 1 (1937) p. 139.