# 27. *Fundamental Theory of Toothed Gearing (V).*

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In the preceding reports——(I) to (IV)——we have explained various properties of profile curves on a plane. Now we shall discuss profile curves on a sphere in this report (V) and in the following reports (VI) and (VII). As well as in the case of plane curves we confine ourselves to deal with such continuous spherical pitch or profile curves as at each of points on them a single tangent may be drawn continuously (although cusps are allowed to exist), and suppose that they make respectively one-point contact motion.

Almost all of the results which we have derived in the case of plane profile curves can be interpreted as the facts on a sphere by replacing a few words, for example——replacing the word "the tangent" on the plane to the word "the tangent great circle" on the sphere.

For the sake of simplicity in the following we shall say merely pitch or profile curves in place of spherical pitch or profile curves, if not needed.

§ i. Necessary and sufficient conditions for profile curves (1).

As a necessary condition that two curves $F_1$ and $F_2$ invariably connected with two pitch curves $K_1$ and $K_2$ respectively be a pair of profile curves we have the following analogue of Descartes theorem for plane profile curves :

($\alpha$). *The common normal great circle to the curves $F_1$ and $F_2$ at any point of contact of them always passes through the common pitch point.*

From the condition ($\alpha$) we obtain the following necessary and sufficient condition for profile curves.

Theorem 1. *A necessary and sufficient condition that two curves $F_1$ and $F_2$ invariably connected with two pitch curves $K_1$ and $K_2$ respectively be a pair of profile curves is that two perpendicular great cireles from any common pitch point to $F_1$ and $F_2$ coincide with each other in the direction and in the arc lngth to their feet.*

We shall say two families of small circles are developable from one upon another, if they consist of circles having centers at corresponding pitch points on $K_1$ and $K_2$ and epual spherical radii. Then we have

Theorem 2. *A necessary and sufficient condition that two curves*

*$F_1$ and $F_2$ invariably connected with two pitch curves $K_1$ and $K_2$ respectively be a pair of profile curves is that they be a pair of suitably chosen envelopes of two families of small circles being developable upon each other having centers on the curves $K_1$ and $K_2$.*

We can prove this theorem by the same process we adopted for the proof of Theorem 2 in the report (I).

From Theorem 2 we can easily derive the following two theorems : (1). *Given three curves $K_\gamma$, $K_1$ and $K_2$ which are all touching at the same one point and starting from this position may roll without sliding along one another. Let $F_\gamma$ and $F_1$ be a pair of profile curves invariably connected with the pitch curves $K_\gamma$ and $K_1$, and similarly $F_\gamma$ and $F_2$ at $K_\gamma$ and $K_2$. Then $F_1$ and $F_2$ are a pair of profile curves having $K_1$ and $K_2$ as a pair of pitch curves.*

(2). *Given three aurves $K_\gamma$, $K_1$ and $K_2$ which are all touching at the same one point and starting from this position may roll without sliding along one another. Let $F_1$ and $F_2$ be the roulettes drawn by the same one point C invariably connected with the curve $K_\gamma$ when $K_\gamma$ makes rolling contact motion along $K_1$ and $K_2$ respectively. Then the curves $F_1$ and $F_2$ are a pair of profile curves having $K_1$ and $K_2$ as a pair of pitch curves.*

As necessary and sufficient conditions that a curve $F$ invariably connected with one $K$ of the pitch curves $K_1$ and $K_2$ be a profile curve, that is, there exists a curve corresponding to $F$ which makes sliding contact motion with $F$, we have the following

Theorem 3. *In order that a curve $F$ invariably connected with one of pitch curves $K_1$ and $K_2$ be a profile curve, each of the following four conditions is respectively necessary and sufficient :*

($\beta$). *The curve $F$ is an envelope of a family of small circles, each of which has its center on the curve $K$ and touch $F$ at one point.*

($\gamma$). *Two normal great circles of the curve $F$ at any two points on it do not pass through the same pitch point.*

($\delta$). *When a point runs on the curve $F$ to a certain direction, the pitch point corresponding to it runs on the curve $K$ also to a definite direction.*

($\varepsilon$). *The curve $F$ is a roulette drawn by a rolling curve and a drawing point suitably defined using $K$ as a base curve.*

Now we can again classify spherical profile curves into those of monotype, namely, positive or negative type, or of mixed type as well as the case of plane profile curves,. In the case that the curve $F$ is particularly of monotype we have the following

Theorem 4. *A necessary and sufficient condition that a curve $F$ invariably connected with a pitch curve $K$ be a profile curve of monotype is that $F$ is such an envelope of a family of small circles with centers on $K$ as each of the circles touchs $F$ at one point and has no common point with $F$ except the point of contact.*

§ 2. Analytical representation of profile curves.

From now on we consider without loss of generality spherical curves on a unit sphere, and assume that the given pitch curve $K$ is oriented to a certain direction, accordingly the length of any arc of $K$ is given a positive or negative sign. The orientation of the tangent great circle to $K$ at any point may be defined in accordance with that of $K$. By the tangent great circle $T$ to $K$ at any point $P$ on $K$, the sphere is divided into two half-spheres. Now take a point $C$ on the sphere. If $C$ exists on the left half-sphere, we give positive sign to the length of the arc of the great circle which connects $P$ with $C$ on the half-sphere. If $C$ exists on the right half-sphere, the length of the arc is negative. Referring to a pair of pitch curves $K_1$ and $K_2$ we shall assume that they are oriented in the same sense, that is, the common tangent great circle at every instant has same sense even if observed as a tangent great circle of $K_1$ or of $K_2$.

Now suppose that a profile curve $F$ is connected invariably with a pitch curve $K$. Take an arbitrary point $P_0$ on $K$ as origin and denote by $P$ such a point on $K$ as the length of arc from $P_0$ to $P$ is $\xi$. Draw the perpendicular great circle to $F$ from $P$ and denote its arc length from $P$ to the foot $C$ on $F$ by $\varphi$ involving its sign. Then we can represent the profile curve $F$ by a relation

$$(1) \qquad \varphi=f(\xi)$$

between $\xi$ and $\varphi$.

When we denote by $\theta$ the angle between the perpendicular great circle $PC$ and the tangent great circle to $K$ at $P$, this angle $\theta$ is in fact determined by the following relations :

$$(2) \qquad \frac{d|\varphi|}{d\xi}=-\cos\theta, \quad sgn(\theta)=sgn(\varphi).$$

If we take arbitrary two corresponding pitch points on a pair of pitch curves $K_1$ and $K_2$ respectively as origins on $K_1$ and $K_2$ then we can represent each of an arbitraily given pair of profile curves by the same equation (1).

The equation of any profile curve $F^*$ parallel to a given profile curve $F$ with Equation (1) is given by

$$(3) \qquad \varphi^*=f^*(\xi)=\begin{cases} f(\xi)+\alpha, & \text{where } |f(\xi)+\alpha|\leqq\pi, \\ f(\xi)+\alpha-sgn(f(\xi)+\alpha)2\pi, & \text{where } |f(\xi)+\alpha|>\pi, \end{cases}$$

where $\alpha$ represents an arbitrary constant.

We may understand Equation (1) of the given profile curve $F$ as the expression giving the length $\varphi$ of the arc of the great circle connecting any point $P$ on $K_\gamma$ with $C$, where $K_\gamma$ means the rolling curve and $C$ the drawing point both of which are determined for

$F$ by Theorem 3. Now we can adopt the epuation $\lambda_r=\lambda_r(\xi)$ as the natural equation of $K_r$ where $\lambda_r$ denotes the spherical radius of curvature of $K_r$. Then we have the following relations :

$$(2)\qquad \frac{d|\varphi|}{d\xi}=-\cos\theta,\quad \operatorname{sgn}(\theta)=\operatorname{sgn}(\varphi)$$

and

$$(4)\qquad \frac{d\theta}{d\xi}=\frac{\sin\theta}{\tan|\varphi|}-\frac{1}{\tan\lambda_r}\,.$$

The above relations are derived from the Stansky's inmovability condition in natural geometry on the sphere.

Transforming (2) and (4) we can derive the natural equation of $K_r$ in the following form :

$$(5)\qquad \lambda_r=\lambda_r(\xi):\quad \tan\lambda_r(\xi)=\frac{\tan f(\xi)\sqrt{1-\{f'(\xi)\}^2}}{1-\{f'(\xi)\}^2-\tan f(\xi)\cdot f''(\xi)}\,.$$

Conversely, if the natural equation $\lambda_r=\lambda_r(\xi)$ of $K_r$ is given at first, then we can derive Equation (1) of the profile curve $F$ corresponding to $K_r$ by solving the differential equation (5) for $f(\xi)$.

§ 3. Necessary and sufficient condition for profile curves (2).

We take the length of arc $\xi$ of a pitch curve $K$ as a variable and consider a one-valued continuous function $\varphi=f(\xi)$ which has a definite sign and differentiable twice in a given range of $\xi$. Then we have

Theorem 5 *In order that a profile curve be given by the equation $\varphi=f(\xi)$, where $f(\xi)$ is a one-valued continuous function which has a definite sign and is differentiable twice with regard to $\xi$, the arc length of a pitch curve, it is necessary and sufficient that the inequality $|f'(\xi)|\leqq 1$ holds in the given interval of $\xi$.*

Expressing Theorem 5 in other words we have

Theorem 6 *Given a family of small circles with centers on a curve $K$ whose spherical radii are given by a function $\varphi=f(\xi)$ one-valued and differentiable twice with regard to $\xi$, the length of arc of $K$. In order that the family possesses an envlope, it is necessary and sufficient that the inequality $|f'(\xi)\leqq 1$ holds in the given interval of $\xi$.*

§ 3. Path of contact.

We can define the path of contact $\Gamma$ of a pair of spherical profile curves by the same process as in the case of plane curves (see the report (II) § 3), and represent $\Gamma$ by a spherical polar equation

$$(6)\qquad \varphi=g(\theta),\quad \operatorname{sgn}(\varphi)=\operatorname{sgn}(\theta)$$

using an arbitrary great circle $T_0$ as initial line and a point $P_0$ on $T_0$ as pole.

The path of contact $\Gamma^*$ of the profile curve $F$ parallel to $F$ with the spherical distance $\alpha$ is given by

$$(7)\qquad \varphi^*=g^*(\theta)=g(\theta)+\alpha,$$

and we have :

*The paths of contact of two parallel profile curves are conchoid curves of each other.*

Now if we give the equation of a profile curve $F$ by (1), Equation (6) of the path of contact $\Gamma$ is derived from (1) and (2) by eliminating $\xi$. Conversely, if Equation (6) of $\Gamma$ is given, we obtain the equation of $F$ eliminating $\theta$ from (2) and (6).

From now on we shall assume, without loss of generality, that $g(\theta)$ is a continuous function with a definite sign. However, the function $g(\theta)$ is not necessarily one-valued as regards $\theta$, although the function $f(\xi)$ is so as regards $\xi$. As to this point the following theorems hold.

Theorem 7. *A necessary and sufficient condition that the func tion $g(\theta)$ which defines a path of contact and has a definite sign is one-valued, continuous and differentiable is that $\theta$ is a conttinuous monotone function of $\xi$ and consequently $\xi$ of $\theta$.*

Theorem 8. *If the function $g(\theta)$ which defines a path of contact and has a definite sign is one-valued, continuous and differentiable, then it is necessary that it is a continuous monotone function in the respective intervals belonging to the quadrant $0<|\theta|<\frac{\pi}{2}$ or $\frac{\pi}{2}<|\theta|<\pi$, and conversely.*

Now suppose a pair of pitch curves are given and besides a curve $\Gamma$——$\varphi=g(\theta)$—— is taken, where $g(\theta)$ is one-valued, continuous and differentiable and has a definite sign. Then the condition that a pair of profile curves with $\Gamma$ as its path of contact may exist is given by the following

Theorem 9. *In order that a path of contact for a pair of profile curves be given by a function $\varphi=g(\theta)$ which is one-valued, continuous and differentiable and has a definite sign, it is necessary and sufficient that $g(\theta)$ be a continuous monotone function in the respective partial intervaals belonging to the quadrant $0<|\theta|<\frac{\pi}{2}$ or $\frac{\pi}{2}<|\theta|<\pi$, and the function $\frac{g'(\theta)}{\cos\theta}$ is integrable in the whole range.*

Now we may understand Equation (6) of path of contact $\Gamma$ as an expression giving the relation between $\varphi$ and $\theta$, in which $\varphi$ is the length of the segment great circle connecting any point $P$ on the rolling curve $K$ determined to the profile curve $F$ with the drawing point $C$, and $\theta$ is the angle between the great circle $PC$ and the tangent great circle to $K_\gamma$ at $P$. Let $\lambda_\gamma$ be the spherical radius of curvature of $K_\gamma$ at $P$, then it holds :

$$\frac{1}{\tan\lambda_\gamma}=\frac{\sin\theta}{\tan|\varphi|}+\frac{\cos\theta}{\frac{d|\varphi|}{d\theta}}. \tag{8}$$

On the other hand, however, the quantity $\lambda_\gamma$ given by (8) is the length measured from the pole $P_0$ to the point $M$ along the great circle $N_0$ drawn passing through $P_0$ perpendicularly to the initial line $T_0$, where $M$ is the point of intersection of $N_0$ and the normal great circle $CM$ drawn to the curve $\Gamma$ at any point $C(\varphi, \theta)$ on it. Hence we have

Theorem 10. *Let $N_0$ be the perpendicular great circle drawn to the initial line $T_0$ at the pole $P_0$. The length of the segment of $N_0$ between $P_0$ and the point $M$ at which the normal great circle to the path of contact $\Gamma$ at any point $C$ on $\Gamma$ intersects with $N_0$ is equal to the spherical radius of curvature of the rolling curve $K_\gamma$ at the pitch point corresponding to $C$.*

Move the rolling curve $K_\gamma$ keeping it to be always touching the great circle $T_0$ at the point $P_0$. Then the drawing point $C$ fixed at $K_\gamma$ at the time runs on the path of contact. Moreover, in this case, the evolute of $K_\gamma$ denoted by $N_\gamma$ makes rolling contact motion along the great circle $N_0$ drawn perpendicularly to $T_0$ passing through $P_0$, in other words: the roulette $\Gamma$ drawn by the point $C$ at the rolling contact motion of the curve $N_\gamma$ along $N_0$ is the very path of contact. Thus we have the following characterization of a profile curve and its path of contact :

*Any profile curve and its path of contact are characterized as the roulette of the same one point which is invariably connected with a suitably taken curve $K_\gamma$ and its evolute $N_\gamma$ when $K_\gamma$ and $N_\gamma$ roll without sliding along the pitch curve $K$ and an arbitrarily determined normal great circle of $K$ respectively.*

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