

37. A Remark on the Goldbach-Vinogradov Theorem.

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(Comm. by Z. SUETUNA, M. J. A., June 16, 1949.)

It was proved in 1937 by Vinogradov ⁽¹⁾ that every odd number from a certain point onwards can be expressed as a sum of three primes, and an asymptotic formula was also given for the number of representations. But by slight modifications of his proof we can extend his result to the following form :

Theorem. *Let k be a positive integer, and let r_1, r_2, r_3 be integers relatively prime to k . Then there exists a number $N_0 = N_0(k) > 3$ depending only on k such that every odd number N greater than N_0 and congruent to $r_1 + r_2 + r_3 \pmod{k}$ can be represented in the form*

$$N = p_1 + p_2 + p_3,$$

where p_n is a prime number congruent to $r_n \pmod{k}$ for $n = 1, 2, 3$.

The number of such representations is expressed for $N > N_0$ asymptotically by

$$(1) \quad \frac{N^2}{2 \log^3 N} \frac{S(N)}{\gamma(N; k)} (1 + \lambda),$$

where

$$S(N) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi^3(q)} \sum_{\substack{(a,q)=1 \\ 0 \leq a < q}} e^{2\pi i \frac{a}{q} N}$$

$\gamma(N; k)$ is the number of solutions (x_1, x_2, x_3) of the congruence

$$x_1 + x_2 + x_3 \equiv N \pmod{k},$$

x_n being an integer such that $0 \leq x_n < k$ and $(k, x_n) = 1$ for each $n = 1, 2, 3$; and further

$$\lambda = \lambda(N; k, r_1, r_2, r_3),$$

$$|\lambda| < K(\log N)^{-\frac{1}{2}} \log \log N,$$

with $K = K(k)$ a positive number depending on k alone.

Let us mention, in passing, that the above function $S(N)$ is known to admit of the expression ⁽²⁾

$$S(N) = \prod_p \left\{ 1 + \frac{1}{(p-1)^3} \right\} \prod_{p|N} \frac{(p-1)(p-2)}{p^2 - 3p + 3},$$

while we can easily deduce the equality

$$\gamma(N; k) = k^2 \prod_{p|(N,k)} \frac{(p-1)(p-2)}{p^2 - 3p + 3} \prod_{p|k} \frac{p^2 - 3p + 3}{p^2},$$

from the multiplicative property of $\gamma(N; k)$, namely

$$\gamma(N; k_1 k_2) = \gamma(N; k_1) \gamma(N; k_2) \text{ if } (k_1, k_2) = 1.$$

Now we shall expound briefly the method of proof. By the aid

of the Lemmas in Vinogradov's paper, we first establish the following theorem.

Theorem 1. *Let h be a constant > 3 ;*

$$\alpha = \frac{a}{q} + \frac{\theta}{q\tau}, \quad (a, q) = 1, \quad |\theta| < 1,$$

$$(\log N)^{3h} \leq q \leq \tau, \quad \tau = N (\log N)^{-3h}$$

Then we have

$$F(r) = \sum_{\substack{p \leq N \\ p \equiv r(h)}} e^{2\pi i \alpha p} \ll N (\log N)^{2-h},$$

where r is a fixed integer such that $(k, r) = 1$.

Now the number $I(N)$ of representations of N in the form

$$N = p_1 + p_2 + p_3, \quad p_n \equiv r_n \pmod{k}, \quad (n=1, 2, 3)$$

can be expressed by the integral

$$I(N) = \int_0^1 F(r_1) F(r_2) F(r_3) e^{-2\pi i \alpha N} d\alpha.$$

Here we change the interval of integration to $[-\frac{1}{f}, 1 - \frac{1}{\tau}]$ and we decompose this interval into major and minor arcs, such that a major arc \mathfrak{M} is an interval of the form $[\frac{a}{q} - \frac{1}{\tau}, \frac{a}{q} + \frac{1}{\tau}]$, $(a, q) = 1$, $0 \leq a < q \leq (\log N)^{3h}$, and the minor arcs \mathfrak{m} are those intervals that remain if we take off all major arcs from the interval $[-\frac{1}{\tau}, 1 - \frac{1}{\tau}]$.

This being so we divide the integral $I(N)$ into two parts $I_1(N)$ and $I_2(N)$ by

$$I_1(N) = \sum_{\mathfrak{M}} \int_{\mathfrak{M}} F(r_1) F(r_2) F(r_3) e^{-2\pi i \alpha N} d\alpha,$$

$$I_2(N) = \sum_{\mathfrak{m}} \int_{\mathfrak{m}} F(r_1) F(r_2) F(r_3) e^{-2\pi i \alpha N} d\alpha,$$

so that we have $I(N) = I_1(N) + I_2(N)$.

Now if α belongs to a minor arc \mathfrak{m} , we must have

$$\alpha = \frac{a}{q} + \frac{\theta}{q\tau}, \quad (a, q) = 1, \quad \theta < 1, \quad (\log N)^{3h} < q \leq \tau,$$

and so we derive, by Theorem 1,

$$(2) \quad I_2(N) \ll N^2 (\log N)^{1-h}.$$

On the other hand, the value of $I_1(N)$ can be calculated, as in Vinogradov, in virtue of the Lemma⁽³⁾

$$\pi(N; q, l) = \frac{1}{\varphi(q)} \int_2^N \frac{dx}{\log x} + O\left(\frac{N(\log N)^{-15h}}{\varphi(q) \log N}\right),$$

$$(0 < q \leq (\log N)^{3h}).$$

The leading term of $I_1(N)$ is found to be

$$\frac{N^2}{2 \log^3 N} \frac{S(N) \cdot}{r(N; k)},$$

and on combining the expression for $I_1(N)$ with the estimation (2), we are led to our formula (1).

References.

- (1) I. Vinogradov : *Some theorems concerning the theory of primes*, Recueil Mathématique, N. S. 2 (1937).
- (2) E. Landau : *Vorlesungen über Zahlentheorie*, Bd. 1, p. 210.
- (3) A. Walfisz : *Zur additiven Zahlentheorie (II)*, Mathematische Zeitschrift 40 (1936).