19. Theory of Invariants in the Geometry of Paths. I. Determination of Covariant Differentiations.

By Tsuneo SUGURI.

Mathematical Institute, 2nd Branch, Kyūsyū University, Kurume.

(Comm. by T. KUBOTA, M.J.A., April 12, 1950.)

§ 0. In an n-dimensional space X_n with a coordinate system x^i $(i=1, 2, \ldots, n)$ let us consider a system of paths of the m-th order defined by

$$(0.1)\qquad x^{(m)i}+H^i(t, x, x^{(1)}, \ldots, x^{(m-1)})=0 \quad (i=1, 2, 3, \ldots, n).$$

The geometries of paths are called ordinary, intrinsic, and rheonomic geometry according to their fundamental transformation groups

$$\text{(i)}\ \begin{cases}\bar{x}^\alpha=\bar{x}^\alpha(x^i),\\ \bar{t}=t,\end{cases}\qquad \text{(ii)}\ \begin{cases}\bar{x}^\alpha=\bar{x}^\alpha(x^i),\\ \bar{t}=\bar{t}(t),\end{cases}\qquad \text{(iii)}\ \begin{cases}\bar{x}^\alpha=\bar{x}^\alpha(t, x^i),\\ \bar{t}=t.\end{cases}$$

Various researches were made already by many geometricians.

In this paper, we study the theory of invariants in the geometry of paths under the so-called generalized rheonomic transformation group

$$(0.2)\qquad \bar{x}^\alpha=\bar{x}^\alpha(t, x^i), \quad \bar{t}=\bar{t}(t)$$

which is a generalization of above three groups; let us assume that the functions $\bar{x}^\alpha(t, x^i)$, $\bar{t}(t)$ have continuous derivatives with respect to $t, x^1, x^2, \ldots, x^n$ up to the order needed, and that none of the functional determinant $\left|\frac{\partial \bar{x}^\alpha}{\partial x^i}\right|$ and the derivative $\frac{d\bar{t}}{dt}$ vanish.

§ 1. Let v and w be two kinds of those geometric objects each of which has uniquely determined components v^i or w_j in every coordinate system x^i in X_n and t and are subject to the transformation law

$$(1.1)\qquad \bar{v}^\alpha=\sigma^p\frac{\partial \bar{x}^\alpha}{\partial x^i}v^i, \quad \bar{w}_\beta=\sigma^p\frac{\partial x^j}{\partial \bar{x}^\beta}w_j, \quad \left(\frac{1}{\sigma}=\frac{d\bar{t}}{dt}\right)$$

under (0.2). We call such a geometric object v or w respectively a *contravariant* or *covariant vector of the second kind of weight* p. A geometric quantity f which is subject to the transformation law $\bar{f}=\sigma^p f$ is called a *scalar of weight* p.

Let us denote the $n+1$ independent variables t, x^i by y^I $(I=0, 1, \ldots, n)$: $y^0\equiv t$, $y^i\equiv x^i$, then the transformation group (0.2) may be written as $\bar{y}^A=\bar{y}^A(y^I)$ in the $X_{n+1}=(t)\times X_n$.

A geometric object v or w, which obeys the ordinary transformation law of vectors in X_{n+1}:

$$(1.2) \qquad \bar{v}^A = \frac{\partial \bar{y}^A}{\partial y^I} v^I, \qquad \bar{w}_B = \frac{\partial y^J}{\partial \bar{y}^B} w_J$$

is called a *vector of the first kind.*

Let the components of a vector v^I or w_J of the first kind be respectively (v^0, v^i) or (w_0, w_j), then we can easily deduce the following facts. (i). *v^0 is a scalar of weight -1.* (ii). *v^i is a contravariant vector of the second kind of weight 0 when and only when $v^0 \equiv 0$.* (i)'. *w_j is a covariant vector of the second kind of weight 0.* (ii)'. *w_0 is a scalar of weight 1 when and only when $w_j \equiv 0$.*

Noticing that there exists a relation

$$\bar{x}^{(\bar{1})\alpha} = \sigma\left(\frac{\partial \bar{x}^\alpha}{\partial x^i} x^{(1)i} + \frac{\partial \bar{x}^\alpha}{\partial t}\right)$$

under (0.2) between the geometric quantities $x^{(1)i} = \dfrac{dx^i}{dt}$ and $\bar{x}^{(\bar{1})\alpha} = \dfrac{d\bar{x}^\alpha}{d\bar{t}}$, we see that (iii). *$V^i = v^i - x^{(1)i} v^0$ is a contravariant vector of the second kind of weight 0,* (iii)'. *$W_J = w_J - y_J^{(1)I} w_I$ is a covariant vector of the first kind whose components are $(-x^{(1)j} w_j, w_j)$,* $y_J^{(1)I}$ being a tensor with components $y_0^{(1)0} \equiv 1$, $y_0^{(1)i} \equiv x^{(1)i}$, $y_j^{(1)I} \equiv 0$.

§2. It is convenient for us to interpret that the system of equations (0.1) gives a correspondence between geometric objects, each formed by a value of t and line-element of the (m-1)-th order $(x^i, x^{(1)i}, \dots, x^{(m-1)i})$, and line-element of the m-th order: $(x^i, x^{(1)i}, \dots, x^{(m-1)i}, x^{(m)i} \equiv -H^i)$. We denote the manifold of these geometric objects $(t, x^i, x^{(1)i}, \dots, x^{(m-1)i})$ by $X_{n+1}^{(m-1)}$. The transformation laws of the line-element of the (m-1)-th order and of functions H^i are given by the following recurrent formulas:

$$(2.1) \quad \begin{cases} \bar{x}^{(\overline{r+1})\alpha} = \sigma \sum\limits_{s=0}^{r} \dfrac{\partial \bar{x}^{(\bar{r})\alpha}}{\partial x^{(s)i}} x^{(s+1)i} + \sigma \dfrac{\partial \bar{x}^{(\bar{r})\alpha}}{\partial t} \quad (r=0, 1, \dots, m-2), \\ -\bar{H}^\alpha = -\sigma \dfrac{\partial \bar{x}^{(\overline{m-1})\alpha}}{\partial x^{(m-1)i}} H^i + \sigma \sum\limits_{s=0}^{m-2} \dfrac{\partial \bar{x}^{(\overline{m-1})\alpha}}{\partial x^{(s)i}} x^{(s+1)i} + \sigma \dfrac{\partial \bar{x}^{(\overline{m-1})\alpha}}{\partial t}. \end{cases}$$

On the other hand a set of differentials $\{dx^{(r)i}\}$ of the line-element of the (m-1)-th order is transformed according to

$$d\bar{x}^{(\bar{r})\alpha} = \sum_{s=0}^{r} \frac{\partial \bar{x}^{(\bar{r})\alpha}}{\partial x^{(s)i}} dx^{(s)i} + \frac{\partial \bar{x}^{(\bar{r})\alpha}}{\partial t} dt \quad (r=0, 1, \dots, m-1),$$

hence if we put

$$(2.2) \quad \begin{cases} \mathfrak{d}x^{(r)i} = dx^{(r)i} - x^{(r+1)i} dt \qquad (r=0, 1, \dots, m-2), \\ \mathfrak{d}x^{(m-1)i} = dx^{(m-1)i} + H^i dt, \end{cases}$$

then these Pfaffians are subject to the transformation

$$\mathfrak{d}\bar{x}^{(r)\alpha}=\sum_{s=0}^{r}\frac{\partial\bar{x}^{(r)\alpha}}{\partial x^{(s)i}}\mathfrak{d}x^{(s)i} \qquad (r=0,1,\dots,m-1). \tag{2.3}$$

This shows that the Pfaffians $\mathfrak{d}x^{(r)i}$ $(r=0, 1, \dots, m-1)$ obey the law analogous to ordinary differentials $dx^{(r)i}$.

From (2.1), we have immediately

$$\frac{\partial\bar{x}^{(r+1)\alpha}}{\partial x^i}=\sigma D_t\frac{\partial\bar{x}^{(r)\alpha}}{\partial x^i} \qquad (r=0,1,\dots,m-2), \tag{2.4}$$

$$\frac{\partial\bar{x}^{(r+1)\alpha}}{\partial x^{(s)i}}=\sigma\left\{D_t\frac{\partial\bar{x}^{(r)\alpha}}{\partial x^{(s)i}}+\frac{\partial\bar{x}^{(r)\alpha}}{\partial x^{(s-1)i}}\right\}\begin{pmatrix}r=1,2,\dots,m-2\\ s=1,2,\dots,r\end{pmatrix}, \tag{2.5}$$

$$\frac{\partial\bar{x}^{(r)\alpha}}{\partial x^{(r)i}}=\sigma^r\frac{\partial\bar{x}^\alpha}{\partial x^i} \qquad (r=0,1,\dots,m-1), \tag{2.6}$$

$$\frac{\partial\bar{x}^{(r)\alpha}}{\partial x^{(r-1)i}}=r\sigma^r D_t\frac{\partial\bar{x}^\alpha}{\partial x^i}+\frac{r(r-1)}{2}\sigma^{r-2}\sigma^{(1)}\frac{\partial\bar{x}^\alpha}{\partial x^i} \tag{2.7}$$

$$(r=1,2,\dots,m-1),$$

where we put $\sigma^{(1)}=\frac{d\sigma}{d\bar{t}}$ and D_t is a scalar operator of weight 1 defined as follows: let f be a differentiable function on $X_{n+1}^{(m-1)}$, then

$$D_t f=\frac{\partial f}{\partial t}+\sum_{r=0}^{m-2}\frac{\partial f}{\partial x^{(r)i}}x^{(r+1)i}-\frac{\partial f}{\partial x^{(m-1)i}}H^i. \tag{2.8}$$

This $D_t f$ is nothing but the derivative of f along the path.

§ **3.** Let v^i be a contravariant vector field of weight p on $X_{n+1}^{(m-1)}$, if we can determine the functions Γ^i_j and Γ such that

$$\boxed{\delta_t v^i=D_t v^i+\Gamma^i_j v^j+p\Gamma v^i} \tag{3.1}$$

is a vector of weight $p+1$, then the vector $\delta_t v^i$ may be seen as a covariant derivative of a vector field v^i along the path. Now we determine the parameters of connection Γ^i_j and Γ by using H^i and their derivatives. Since $\delta_t v^i$ is a vector of weight $p+1$, so Γ^i_j and Γ are transformed as

$$\frac{\partial\bar{x}^\beta}{\partial x^j}\bar{\Gamma}^\alpha_\beta=\sigma\frac{\partial\bar{x}^\alpha}{\partial x^i}\Gamma^i_j+\sigma D_t\frac{\partial\bar{x}^\alpha}{\partial x^j}, \tag{3.2}$$

$$\sigma\bar{\Gamma}=\sigma^2\Gamma-\sigma^{(1)}. \tag{3.3}$$

Noticing (2.1), (2.6), (2.7), we can see that the functions defined by

$$\boxed{\Gamma^i_j=G^{ik}\frac{\partial^3 H^l}{\partial x^{(m-1)l}\,\partial x^{(m-1)k}\,\partial x^{(m-2)j}}-\frac{m-2}{m}\frac{\partial H^i}{\partial x^{(m-1)j}},} \tag{3.4}$$

$$\boxed{\Gamma=\frac{1}{n}\left\{\frac{2}{m}\frac{\partial H^i}{\partial x^{(m-1)i}}-\frac{2}{m-1}G^{ij}\frac{\partial^3 H^l}{\partial x^{(m-1)l}\,\partial x^{(m-1)j}\,\partial x^{(m-2)i}}\right\}} \tag{3.5}$$

behave as requested. G^{ij} in (3.4), (3.5) is a contravariant tensor of weight $2m-3$ and defined in the following manner together with covariant tensor G_{ij} of weight $-(2m-3)$:

$$G_{ij}=\frac{\partial^3 H^l}{\partial x^{(m-1)l}\,\partial x^{(m-1)j}\,\partial x^{(m-1)i}},\quad G^{ij}G_{jk}=\delta^i_k,\quad |G_{ij}|\neq 0.$$

§4. To derive a covariant differential of the line-element, let us begin with the proof of the

Theorem. *If Pfaffians* $P^i(d)=\sum_{r=0}^{M} P^i_{(r)k}\,\mathfrak{d}x^{(r)k}$ *are transformed as components of a contravariant vector of weight* p, *then the Pfaffians*

$$\delta_t P^i(d)=\sum_{r=0}^{M} P^i_{(r)k}\,\mathfrak{d}x^{(r+1)k}+\sum_{r=0}^{M}\{D_t P^i_{(r)k}+\Gamma^i_j P^j_{(r)k}+p\Gamma P^i_{(r)k}\}\,\mathfrak{d}x^{(r)k}$$

are also transformed as components of a contravariant vector of weight $p+1$.

By virtue of (2.3), (3.2) and (3.3) this can be proved without great difficulty.

It is evident from (2.3) that $\delta x^i\equiv\mathfrak{d}x^i$ $(i=1,2,\dots,n)$ are Pfaffians with vector character of weight 0, hence by virtue of the theorem we see that $\delta x^{(r+1)i}\equiv\delta_t(\delta x^{(r)i})$ $(r=0,1,\dots,m-2)$ are Pfaffians with vector charactor of weight $r+1$. Therefore we may use these $\delta x^{(r)i}$ $(r=0,1,\dots,m-1)$ as desired covariant differentials of the line-element. They will be represented explicitely in terms of $\mathfrak{d}x^{(r)i}$ as

$$(4.1)\quad \begin{cases}\delta x^i=\mathfrak{d}x^i,\\ \delta x^{(r)i}=\mathfrak{d}x^{(r)i}+\sum_{s=0}^{r-1}\Lambda^{(r)i}_{(s)j}\,\mathfrak{d}x^{(s)j} & (r=1,2,\dots,m-1);\end{cases}$$

then $\Lambda^{(r)i}_{(s)j}$ are defined by the following recurrent formulas:

$$(4.2)\quad \begin{cases}\Lambda^{(r+1)i}_{(r)j}=(r+1)\,\Gamma^i_j+\dfrac{r(r+1)}{2}\Gamma\delta^i_j & (r=0,1,\dots,m-2),\\ \Lambda^{(r+1)i}_{(s)j}=D_t\Lambda^{(r)i}_{(s)j}+\Lambda^{(r)i}_{(s-1)j}+(\Gamma^i_h+r\Gamma\delta^i_h)\Lambda^{(r)h}_{(s)j} & \begin{pmatrix}r=2,3,\dots,m-2\\ s=1,2,\dots,\ r-1\end{pmatrix},\\ \Lambda^{(r+1)i}_{(0)j}=D_t\Lambda^{(r)i}_{(0)j}+(\Gamma^i_h+r\Gamma\delta^i_h)\Lambda^{(r)h}_{(0)j} & (r=1,2,\dots,m-2).\end{cases}$$

(4.1) can be solved in $\mathfrak{d}x^{(r)i}$ as

$$(4.3)\quad \begin{cases}\mathfrak{d}x^i=\delta x^i,\\ \mathfrak{d}x^{(r)i}=\delta x^{(r)i}-\sum_{s=0}^{r-1}\Omega^{(r)i}_{(s)j}\,\delta x^{(s)j} & (r=1,2,\dots,m-1),\end{cases}$$

where

$$(4.4)\quad \begin{cases}\Omega^{(r)i}_{(r-1)j}=\Lambda^{(r)i}_{(r-1)j} & (r=1,2,\dots,m-1),\\ \Omega^{(r)i}_{(s)j}=\Lambda^{(r)i}_{(s)j}-\sum_{t=s+1}^{r-1}\Lambda^{(r)i}_{(t)h}\,\Omega^{(t)h}_{(s)j} & \begin{pmatrix}r=2,3,\dots,m-1\\ s=0,1,\dots,\ r-2\end{pmatrix}.\end{cases}$$

The transformation laws of $\Omega^{(r)i}_{(s)j}$ and $\Lambda^{(r)i}_{(s)j}$ are given respectively as follows:

$$\frac{\partial \bar{x}^{(\bar{r})\alpha}}{\partial x^{(s)j}}=\sum_{t=s+1}^{r}\frac{\partial \bar{x}^{(\bar{r})\alpha}}{\partial x^{(t)i}}\Omega^{(t)i}_{(s)j}-\sigma^{s}\frac{\partial \bar{x}^{\beta}}{\partial x^{j}}\bar{\Omega}^{(\bar{r})\alpha}_{(\bar{s})\beta}\quad \begin{pmatrix} r=1,2,\dots,m-1\\ s=0,1,\dots,\ r-1\end{pmatrix}, \tag{4.5}$$

$$\frac{\partial \bar{x}^{(\bar{r})\alpha}}{\partial x^{(s)j}}=\sigma^{r}\frac{\partial \bar{x}^{\alpha}}{\partial x^{i}}\Lambda^{(r)i}_{(s)j}-\sum_{t=s}^{r-1}\frac{\partial \bar{x}^{(\bar{t})\beta}}{\partial x^{(s)j}}\bar{\Lambda}^{(\bar{r})\alpha}_{(\bar{t})\beta}\quad \begin{pmatrix} r=1,2,\dots,m-1\\ s=0,1,\dots,\ r-1\end{pmatrix}. \tag{4.6}$$

But we notice that they are equivalent to each other.

§ **5.** Now let us determine the covariant differential Dv^i of a vector field v^i in $X^{(m-1)}_{n+1}$ of weight p as follows:

$$\boxed{Dv^i=dv^i+(\Gamma^i_j v^j+p\Gamma v^i)dt+\Gamma^i_{jk}v^j\mathfrak{d}x^k.} \tag{5.1}$$

Making use of (3.2) and (3.3), we see that the unknown parameters of connection Γ^i_{jk} are subject to the transformation

$$\frac{\partial \bar{x}^{\alpha}}{\partial x^{i}}\Gamma^i_{jk}=\frac{\partial \bar{x}^{\beta}}{\partial x^{j}}\frac{\partial \bar{x}^{\gamma}}{\partial x^{k}}\bar{\Gamma}^{\alpha}_{\beta\gamma}+\frac{\partial^2 \bar{x}^{\alpha}}{\partial x^{j}\partial x^{k}}. \tag{5.2}$$

By virtue of (3.2) or (2.1) and (4.5), Γ^i_{jk} can be defined by

$$\boxed{\Gamma^i_{jk}=\frac{\partial \Gamma^i_j}{\partial x^{(1)k}}-\sum_{s=2}^{m-1}\Omega^{(s)h}_{(1)k}\frac{\partial \Gamma^i_j}{\partial x^{(s)h}},} \tag{5.3}$$

or

$$\boxed{\Gamma^i_{jk}=\frac{1}{m}\left\{\frac{\partial^2 H^i}{\partial x^{(m-1)j}\partial x^{(1)k}}-\sum_{s=2}^{m-1}\Omega^{(s)h}_{(1)k}\frac{\partial^2 H^i}{\partial x^{(m-1)j}\partial x^{(s)h}}\right\}.} \tag{5.4}$$

Then we obtain the covariant derivatives of the second kind of a vector field v^i of weight p by decomposing its covariant differential in terms of dt, $\delta x^{(r)i}$ $(r=0, 1, \dots, m-1)$ as in the usual way:

$$Dv^i=(\nabla v^i)dt+\sum_{r=0}^{m-1}(\nabla_{(r)j}v^i)\delta x^{(r)j} \tag{5.5}$$

and

$$\begin{cases} \nabla v^i=D_t v^i+\Gamma^i_j v^j+p\Gamma v^i,\\ \nabla_{(m-1)j}v^i=\dfrac{\partial v^i}{\partial x^{(m-1)j}},\\ \nabla_{(r)j}v^i=\dfrac{\partial v^i}{\partial x^{(r)j}}-\displaystyle\sum_{s=r+1}^{m-1}\Omega^{(s)h}_{(r)j}\dfrac{\partial v^i}{\partial x^{(s)h}}\qquad (r=1,\ 2,\ \dots,\ m-2),\\ \nabla_{(0)j}v^i=\dfrac{\partial v^i}{\partial x^{j}}+\Gamma^i_{kj}v^k-\displaystyle\sum_{s=1}^{m-1}\Omega^{(s)h}_{(0)j}\dfrac{\partial v^i}{\partial x^{(s)h}}. \end{cases} \tag{5.6}$$

We must notice that the covariant derivatives ∇v^i, $\nabla_{(r)j}v^i$ of a vector field v^i of weight p is respectively of weight $p+1$, $p-r$.

§ 6. The differential invariants of the connection defined in the preceding paragraph may be easily obtained. They are the curvature tensors of the second kind $R^i_{h(r)j}$, $R_{(r)j}$, $R^i_{h(r)j(0)k}$ $(r=0, 1, \dots, m-1)$; torsion tensors of the second kind $S^{(m-1)h}_{(r)j}$ $(r=0, 1, \dots, m-1)$, $S^{(t)h}_{(r)j(s)k}$ $(r=0, 1, \dots, m-1;\ s=0, 1, \dots, r;\ s\neq m-1;\ t=s+1, s+2, \dots, m-1;$ $t=0$ is allowed when and only when $r=s=0)$ and their successive covariant derivatives. Curvature tensors and torsion tensors may be obtained geometrically by parallel displacement of a vector along the closed infinitesimal circuit in $X^{(m-1)}_{n+1}$ or analytically by construction of all the commutators of the operators ∇ and $\nabla_{(r)j}$ $(r=0, 1, \dots, m-1)$.

§ 7. Let us determine the covariant differential D^*v^I of a vector field v^I of the first kind in $X^{(m-1)}_{n+1}$ by the following manner:

$$\boxed{D^*v^I = dv^I + {}^*\Gamma^I_{JK} v^J dy^K.} \tag{7.1}$$

The unknown parameters of connection ${}^*\Gamma^I_{JK}$ are transformed in the same manner as the parameters of usual affine connection. Therefore noticing the transformation law we can determine the ${}^*\Gamma^I_{JK}$ in terms of the functions H^i and their derivatives, however we determine them geometrically basing on the relations mentioned in § 1 between vector fields of the first and the second kind as follows.

(i). Let the components of a vector field v^I be (v^0, v^i), then v^0 is a scalar field of weight -1, hence under the geometrical condition

$$\boxed{(D^*v^I)_{I=0} \equiv Dv^0,} \tag{7.2}$$

we have

$$\boxed{{}^*\Gamma^0_{0k} = {}^*\Gamma^0_{j0} = {}^*\Gamma^0_{jk} = 0, \quad {}^*\Gamma^0_{00} = -\Gamma.} \tag{7.3}$$

(ii). As we know that $v^i - x^{(1)i}v^0$ is a vector field of weight 0, setting the geometrical condition

$$\boxed{(D^*v^I)_{I=i} - x^{(1)i}(D^*v^I)_{I=0} \equiv D(v^i - x^{(1)i}v^0) + v^0\delta x^{(1)i},} \tag{7.4}$$

and noticing (7.3) we have

$$\boxed{\begin{aligned} &{}^*\Gamma^i_{jk} = \Gamma^i_{jk}, \\ &{}^*\Gamma^i_{j0} = \Gamma^i_j - \Gamma^i_{jk}x^{(1)k}, \\ &{}^*\Gamma^i_{0k} = \Gamma^i_k - \Gamma^i_{jk}x^{(1)j}, \\ &{}^*\Gamma^i_{00} = -\{x^{(2)i} + 2\Gamma^i_j x^{(1)j} + \Gamma x^{(1)i} - \Gamma^i_{jk}x^{(1)j}x^{(1)k}\}. \end{aligned}} \tag{7.5}$$

Thus we have determined all the parameters of connection.

(iii). v^i is a contravariant vector field of weight 0 of the second kind when and only when $v^0 \equiv 0$. We know that the following geometrical relations are identically satisfied on account of (i) and (ii):

$$(7.6) \qquad \boxed{(D^*v^I)_{I=0} \equiv 0, \ (D^*v^I)_{I=i} \equiv Dv^i, \quad \text{when} \quad v^0 \equiv 0.}$$

These results may also be obtained by using a covariant vector field instead of a contravariant vector field in somewhat different manner.

The covariant derivatives of a vector field v^I are obtainable by decomposing its covariant differential (7.1) in terms of dt and $\delta x^{(r)i}$:

$$(7.7) \qquad D^*v^I = (\nabla^* v^I)\, dt + \sum_{r=0}^{m-1} (\nabla^*_{(r)j} v^I)\, \delta x^{(r)j}.$$

$\nabla^* v^I$, $\nabla^*_{(r)j} v^I$ $(r=0, 1, \dots, m-1)$ are the covariant derivatives of the first kind.

The differential invariants of the connection are the curvature tensors of the first kind $R^{*I}_{H(r)j}$, $R^{*I}_{H(r)j(0)k}$ $(r=0, 1, \dots, m-1)$; torsion tensors of the first kind $S^{*(m-1)h}_{(r)j}$ $(r=0, 1, \dots, \mathrm{m}-1)$, $S^{*(t)h}_{(r)j(s)k}$ $(r=0, 1, \dots, m-1;\ s=0, 1, \dots, r;\ s \neq m-1;\ t=s+1, s+2, \dots, m-1;\ t=0$ is allowed when and only when $r=s=0)$ and their successive covariant derivatives.

The relations between covariant derivatives of the first kind and the second kind and between curvature and torsion tensors of the two kinds are obtainable by using the geometrical conditions (7.2), (7.4) and (7.6). *The torsion tensors of the first kind coincide identically with the torsion tensors of the second kind. The curvature tensors of the first kind are expressible by the curvature tensors of the second kind and the torsion tensors* $S^{(1)h}_{(r)j(0)k}$ $(r=0, 1, \dots, m-1)$.