23. Wiman's Theorem on Integral Functions of Order $<\frac{1}{2}$.

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(Comm. by T. TAKAGI, M.J.A., May 12, 1950.)

1. Density of sets.

Let E be a measurable set on the positive x-axis and $E(a,b)$ be its part contained in $[a,b]$. We put

$$\bar{\delta}(E)=\varlimsup_{r\to\infty}\frac{1}{r}\int_{E(0,r)}dr \qquad , \quad \underline{\delta}(E)=\varliminf_{r\to\infty}\frac{1}{r}\int_{E(0,r)}dr \qquad , \tag{1}$$

$$\bar{\lambda}(E)=\varlimsup_{r\to\infty}\frac{1}{\log r}\int_{E(1,r)}\frac{dr}{r} \qquad , \quad \underline{\lambda}(E)=\varliminf_{r\to\infty}\frac{1}{\log r}\int_{E(1,r)}\frac{dr}{r} \qquad , \tag{2}$$

$$\bar{\lambda}^*(E)=\varlimsup_{r/a\to\infty}\frac{1}{\log(r/a)}\int_{E(a,r)}\frac{dr}{r}, \quad \underline{\lambda}^*(E)=\varliminf_{r/a\to\infty}\frac{1}{\log(r/a)}\int_{E(a,r)}\frac{dr}{r} \quad (a\geqq 1). \tag{3}$$

We call (1) the upper (lower) density, (2) the upper (lower) logarithmic density and (3) the upper (lower) strong logarithmic density. Evidently

$$0\leqq\underline{\delta}(E)\leqq\bar{\delta}(E)\leqq 1, \quad 0\leqq\underline{\lambda}^*(E)\leqq\underline{\lambda}(E)\leqq\bar{\lambda}(E)\leqq\bar{\lambda}^*(E)\leqq 1$$

and

$$\underline{\delta}(E)+\bar{\delta}(C(E))=1, \quad \underline{\lambda}(E)+\bar{\lambda}(C(E))=1, \quad \underline{\lambda}^*(E)+\bar{\lambda}^*(C(E))=1,$$

where $C(E)$ is the complementary set of E. We shall prove:

Lemma 1. $0\leqq\underline{\delta}(E)\leqq\underline{\lambda}^*(E)\leqq\underline{\lambda}(E)\leqq\bar{\lambda}(E)\leqq\bar{\lambda}^*(E)\leqq\bar{\delta}(E)\leqq 1.$

Proof. Let $\bar{\delta}(E)=\alpha$, then for any $\varepsilon>0$,

$$\mu(r)=\int_{E(0,r)}dr\leqq r(\alpha+\varepsilon) \qquad (r\geqq r_0(\varepsilon)>1),$$

so that if $1\leqq a<r_0<r$, since $\mu(r)\leqq r$,

$$\int_{E(a,r)}\frac{dr}{r}\leqq\int_1^{r_0}\frac{dr}{r}+\int_{r_0}^{r}\frac{d\mu(r)}{r}$$

$$\leqq r_0+\left[\frac{\mu(r)}{r}\right]_{r_0}^{r}+\int_{r_0}^{r}\frac{\mu(r)}{r^2}dr\leqq r_0+1+(\alpha+\varepsilon)\int_{r_0}^{r}\frac{dr}{r}$$

$$\leqq r_0+1+(\alpha+\varepsilon)\log\frac{r}{a}.$$

If $r_0\leqq a<r$, then similarly

$$\int_{E(a,r)}\frac{dr}{r}\leqq 1+(\alpha+\varepsilon)\log\frac{r}{a}.$$

From this we have

$$\bar{\lambda}^*(E) \leqq \alpha = \bar{\delta}(E).$$

Similarly, we can prove $\underline{\delta}(E) \leqq \bar{\lambda}^*(E)$, q.e.d.

2. Main theorem.

Let $f(z)$ be an integral function of order $\rho(0<\rho<\frac{1}{2})$ and

$$m(r) = \underset{|z|=r}{\mathrm{Min.}} |f(z)|, \qquad M(r) = \underset{|z|=r}{\mathrm{Max.}} |f(z)|.$$

Then Wiman proved that there exists $r_n \to \infty$, such that $m(r_n) \to \infty$. Besicovitch and Pennycuick[1)] proved that

$$\bar{\delta}[E(\log m(r) > r^{\rho-\varepsilon})] \geqq 1-2\rho \text{ for any } \varepsilon > 0 \tag{4}$$

and that there exists an integral function of any order $\rho(0<\rho<1)$, such that

$$\underline{\delta}[E(\log m(r) > -r^{\rho-\varepsilon})] = 0 \quad \text{for any} \quad \varepsilon > 0, \tag{5}$$

where $E(\log m(r) > \alpha)$ is the set of r, such that $\log m(r) > \alpha$.

We shall prove

Theorem 1. (*Main theorem*). *Let $f(z)$ be an integral function of order $\rho(0<\rho<\frac{1}{2})$, then*

(i) $$\bar{\lambda}^*[E(\log m(r) > r^{\rho-\varepsilon})] \geqq 1-2\rho$$

for any $\varepsilon > 0$.

(ii) *If* $\overline{\lim\limits_{r\to\infty}} \dfrac{\log M(r)}{r^\rho} = \infty$, *then*

$$\bar{\lambda}^*[E(\log m(r) > kr^\rho)] \geqq 1-2\rho$$

for any $k > 0$.

(iii) *There exists an integral function of any order $\rho(0<\rho<\frac{1}{2})$, such that*

$$\bar{\lambda}[E(\log m(r) > r^{\rho-\varepsilon})] < 1-2\rho, \quad 0<\varepsilon<\rho(1-2\rho).$$

(4) follows from (i) by Lemma 1.

3. Some lemmas.

Let D be a domain on the z-plane, which contains $z=0$ and $z=\infty$ belongs to its boundary Δ. Let D_r be the part of D, which is contained in $|z|<r$. Then D_r consists of at most a countable number of connected domains. Let D_r^0 be the connected one, which contains $z=0$ and θ_r be the part of the boundary of D_r^0, which lies on $|z|=r$.

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1) A. S. Besicovitch: On integral functions of order<1. Math. Ann. **97** (1927). Besicovitch's proof is valid only for functions of regular growth. The general case was proved by K. Pennycuick: On a theorem of Besicovitch: Jour. London Math. Soc. **10** (1935). Bokjellberg: On certain integral and harmonic functions. Thèse. Upsala (1984). M. Inoue: Sur le module minimum des fonctions sousharmoniques et des fonctions entières d'ordre<$\frac{1}{2}$. Mem. Fac. Sci. Kyusyu Univ. ser. A. Vol. IV. No. 2 (1949).

Then θ_r consists of at most a countable number of arcs $\{\theta_r^{(i)}\}$ and let $r\theta(r)$ be the maximum of lengths of these arcs. We define $\bar{\theta}(r)$ as follows. If $|z|=r$ meets Λ, then we put $\bar{\theta}(r)=\theta(r)$ and if $|z|=r$ does not meet Λ and is contained entirely in D, then we put $\bar{\theta}(r)=\infty$. Let $u_r(z)$ be a harmonic function in D_r^0, such that $u_r(z)=0$ at regular points on the boundary at D_r^0, which lies in $|z|<r$ and $u_r(z)=1$ on θ_r. Then $u_r(z)$ is the harmonic measure of θ_r with respect to D_r^0. I have proved in the former paper[2] that

$$u_r(z) \leqq \text{const. } e^{-\pi\int_{2|z|}^{\frac{r}{2}} \frac{dr}{r\bar{\theta}(r)}}, \tag{6}$$

where const. is a pure numerical constant.
Let E be the set of r, such that $|z|=r$ meets Λ, then $\bar{\theta}(r)\leqq 2\pi$ for $r\in E$ and $\bar{\theta}(r)=\infty$ otherwise, so that

Lemma 2.

$$u_r(z) \leqq \text{const. } e^{-\frac{1}{2}\int_{E(2|z|,\, r/2)} \frac{dr}{r}} \qquad (r>4\,|z|).$$

Beurling[3] proved that

$$u_r(z) \leqq 2e^{-\frac{1}{2}\int_{E(|z|,\, r)} \frac{dr}{r}}, \tag{7}$$

but since we shall use (6) latter and Lemma 2 suffices for the later proof, we use Lemma 2 instead of (7).

Lemma 3. *Let E be a closed set on the positive real axis of the z-plane, such that*

$$\underline{\lambda}^*(E) > \alpha$$

and $u_r(z)$ be a harmonic function in $|z|<r$, except on $E(0, r)$, such that $u_r(z)=0$ on $E(0, r)$ at its regular points and $u_r(z)=1$ on $|z|=r$. Then

$$u_r(z) \leqq \text{const.} \left(\frac{|z|}{r}\right)^{\frac{\alpha}{2}}, \quad \textit{if } r \geqq k_0|z|,\ |z|\geqq 1,$$

where k_0 is a certain constant (>1).

Proof. Since $\underline{\lambda}^*(E)>\alpha$, we have if $\frac{r}{|z|}\geqq k_0$,

$$\int_{E(2|z|,\, r/2)} \frac{dr}{r} \geqq \alpha \log \frac{r}{|z|} \qquad (|z|\geqq 1),$$

so that by Lemma 2,

2) M. Tsuji: A theorem on tbe majoration of harmonic measure and its applications. Tohoku Math. Jour. **3** (1951).

3) Beurling: Etudes sur un problème de majoration. Thèse Upsala (1933). M. Inoue: Une ètude sur les fonctions sousharmoniques et ses applications aux fonctions holomorphes. Mem. Fac. Sci. Kyusyu Univ. (1943).

$$u_r(z) \leqq \text{const.} \left(\frac{|z|}{r}\right)^{\frac{\alpha}{2}} \quad (r \geqq k_0|z|,\ |z| \geqq 1).$$

Lemma 4. *Let E be a closed set on the positive real axis of the z-plane, such that*

$$\underline{\lambda}^*(E) > 2k \quad (0 < k < \tfrac{1}{2}).$$

Then there exists a harmonic function $u(z) > 0$ outside E, such that $u(r) = r^k$ on E at its regular points and

$$0 < u(z) \leqq \text{const.}\, |z|^k \quad (|z| \geqq 1).$$

Proof. Let $u_r(z)$ be the harmonic function defined by Lemma 3 and $v_r(z)$ be a harmonic function outside E, such that $v_r(z) = 0$ on $E(0, r)$ and $v_r(z) = 1$ on $E - E(0, r)$ at its regular points. Then

$$v_r(z) \leqq u_r(z) \quad \text{in} \quad |z| < r.$$

We take k_1, such that

$$\underline{\lambda}^*(E) > 2k_1 > 2k \qquad (k_1 > k),$$

then by Lemma 3,

$$v_r(z) \leqq u_r(z) \leqq \text{const.} \left(\frac{|z|}{r}\right)^{k_1} \quad (r \geqq k_0|z|,\ |z| \geqq 1), \tag{8}$$

so that the integral

$$u(z) = k\int_0^\infty v_r(z)\, r^{k-1}\, dr \text{ [4]} \tag{9}$$

converges and represents a harmonic function outside E.
Let $z = r_0$ be a regular point of E, then if z tends to r_0 from the outside of E, then $\lim\limits_{z \to r_0} v_r(z) = v_r(r_0)$.
Since $v_r(z)$ is majorated by (8), we have by Lebesgue's theorem,

$$\lim_{z \to r_0} u(z) = k\int_0^\infty v_r(r_0) r^{k-1}\, dr = k\int_0^{r_0} r^{k-1}\, dr = r_0^k, \tag{10}$$

so that $u(r) = r^k$ on E at its regular points.

Since $0 \leqq v_r(z) \leqq 1$, we have from (9),

$$u(z) \leqq k\int_0^{k_0|z|} r^{k-1}\, dr + k\int_{k_0|z|}^\infty v_r(z)\, r^{k-1}\, dr$$

$$\leqq (k_0|z|)^k + \text{const.} \int_{k_0|z|}^\infty \left(\frac{|z|}{r}\right)^{k_1} r^{k-1}\, dr = (k_0|z|)^k + \text{const.}\, |z|^k \int_{k_0}^\infty \frac{dt}{t^{1+k_1-k}}$$

$$(r = |z|\, t)$$

$$\leqq \text{const.}\, |z|^k.$$

4. Proof of the main theorem.

Let

$$f(z) = \prod_{n=1}^\infty \left(1 - \frac{z}{a_n}\right)$$

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4) The expression of $u(z)$ in the form (9) and the proof of (10) are suggested to the author by A. Mori.

be an integral function of order $\rho(0<\rho<\frac{1}{2})$, then since

$$\prod_{n=1}^{\infty}\left|1-\frac{r}{|a_n|}\right|\leqq m(r)\leqq M(r)\leqq\prod_{n=1}^{\infty}\left(1+\frac{r}{|a_n|}\right),$$

we may suppose, for the proof, that all a_n are positive, so that

$$m(r)=\prod_{n=1}^{\infty}\left|1-\frac{r}{a_n}\right|=|f(r)|,\ M(r)=\prod_{n=1}^{\infty}\left(1+\frac{r}{a_n}\right)=f(-r),\ (a_n>0).$$

(i) Let

$$E=E\,(\log m(r)\leqq r^{\rho_1})\quad(\rho_1=\rho-\varepsilon) \tag{11}$$

and suppose that

$$\underline{\lambda}^*(E)>2\rho\,(>2\rho_1), \tag{12}$$

so that

$$\underline{\lambda}(E)\geqq\underline{\lambda}^*(E)>2\rho\,. \tag{13}$$

We construct a harmonic function $u(z)$ by Lemma 4, with $k=\rho_1$, such that $u(r)=r^{\rho_1}$ on E at its regular points, then

$$u(-R)\leqq \text{const. } R^{\rho_1}\ (R\geqq 1).$$

Since $\rho_1<\rho$, there exists R_0, such that

$$\log M(R_0)-u(-R_0)=\log|f(-R_0)|-u(-R_0)>0.$$

Let $u_r(z)$ be defined as in Lemma 3, then since

$$\log|f(r)|-u(r)=\log m(r)-r^{\rho_1}\leqq 0 \text{ on } E,$$

we have

$$\log|f(z)|-u(z)\leqq\log M(r)u_r(z) \text{ in } |z|<r,$$

so that by Lemma 2,

$$0<\log|f(-R_0)|-u(-R_0)\leqq\log M(r)u_r(-R_0)$$

$$\leqq \text{const. } \log M(r)\,e^{-\frac{1}{2}\int_{E(2R_0,\,r/2)}\frac{dr}{r}}\qquad(r>4R_0),$$

hence

$$\frac{1}{2}\int_{E(2R_0,\,r/2)}\frac{dr}{r}\leqq \text{const.}+\log\log M(r).$$

From this we have

$$\bar{\lambda}(E)=\overline{\lim_{r\to\infty}}\frac{1}{\log r}\int_{E(1,\,r)}\frac{dr}{r}\leqq 2\rho,$$

which contradicts (13). Hence $\underline{\lambda}^*(E)\leqq 2\rho$, so that

$$\bar{\lambda}^*[(E\,(\log m(r)>r^{\rho-\varepsilon})]\geqq 1-2\rho. \tag{14}$$

(ii) Suppose that

$$\overline{\lim_{r\to\infty}}\frac{\log M(r)}{r^{\rho}}=\infty \tag{15}$$

and let

$$E=E(\log m(r)\leqq kr^{\rho})\qquad(k>0) \tag{16}$$

and suppose that $\underline{\lambda}^*(E)>2\rho$.

Then we construct a harmonic function $u(z)$ by Lemma 4, such that $u(r)=kr^{\rho}$ on E at its regular points, then by Lemma 4,

$$u(-R)\leqq \text{const. } R^{\rho}.$$

By (15), there exists R_0, such that

$$\log M(R_0)-u(-R_0)=\log|f(-R_0)|-u(-R_0)>0.$$

From this we proceed similarly as in (i) and we can prove that

$$\bar{\lambda}^*[E(\log m(r)>kr^{\rho})]\geqq 1-2\rho \quad \text{for any} \quad k>0.$$

(iii) Next we shall prove that there exists an integral function of order $\rho(0<\rho<\frac{1}{2})$, such that

$$\bar{\lambda}[E(\log m(r)>r^{\rho-\varepsilon})]<1-2\rho, \quad (0<\varepsilon<\rho(1-2\rho)).$$

Since $0<\varepsilon<\rho(1-2\rho)$,

$$\rho-\varepsilon>2\rho^2>0, \quad \frac{1-2\rho}{2\rho}>\frac{\varepsilon}{\rho-\varepsilon}.$$

We choose δ, such that $\dfrac{1-2\rho}{2\rho}>\delta>\dfrac{\varepsilon}{\rho-\varepsilon}$, then

$$\frac{\delta}{1+\delta}<1-2\rho, \quad \frac{1}{\rho}(1+\delta)(\rho-\varepsilon)=1+s \quad (s>0). \tag{17}$$

Let

$$n_{i+1}=\left[e^{n_i^{\frac{s}{2}}}\right] \quad (i=1,2,\ldots), \tag{18}$$

where $[x]$ is the integral part of x and we choose n_1 so large that $1<n_1<n_2<\cdots<n_i\to\infty$.

Let P_i be a point on the curve $y=x^k$ $(k=\frac{1}{\rho}>1)$, whose $x=n_i$.

We connect P_i, P_{i+1} by a rectilenear segment L_i, whose equation is

$$y=a_ix-\beta_i, \tag{19}$$

where

$$a_i=\frac{n_{i+1}^k-n_i^k}{n_{i+1}-n_i}\sim n_{i+1}^{k-1}, \quad \beta_i=\frac{n_{i+1}^k n_i-n_i^k n_{i+1}}{n_{i+1}-n_i}, \tag{20}$$

so that

$$\frac{\beta_i}{a_i}=n_i-\eta_i \quad (\eta_i>0), \tag{21}$$

$$\eta_i=\frac{n_i^k(n_{i+1}-n_i)}{n_{i+1}^k-n_i^k}\sim\frac{n_i^k}{n_{i+1}^{k-1}}\to 0 \quad (i\to\infty). \tag{22}$$

By (19), $n_i\leqq x\leqq n_{i+1}$ is mapped on $n_i^k\leqq y\leqq n_{i+1}^k$. Let

$$f(z)=\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right), \tag{23}$$

where $\qquad a_n=a_in-\beta_i \quad (n_i\leqq n\leqq n_{i+1}).$

The curve, which is composed of L_i $(i=1,2,\ldots)$ is called the curve of roots of $f(z)$ in Besicovitch's paper.

Since L_i lies above the curve $y=x^k$,

$$a_n \geqq n^k \quad (n=1, 2, \dots), \quad a_{n_i}=n_i^k \quad (i=1, 2, \dots),$$

so that the convergence exponent of a_n is $\frac{1}{k}=\rho$, hence $f(z)$ is an integral function of order ρ and

$$m(r)=\prod_{n=1}^{\infty}\left|1-\frac{r}{a_n}\right| . \tag{24}$$

We shall prove that

$$\bar{\lambda}[E(\log m(r) > r^{\rho-\varepsilon})] \leqq \frac{\delta}{1+\delta} < 1-2\rho.$$

Let $n_i^k \leqq r \leqq n_{i+1}^k$, then

$$r=\alpha_i\tau-\beta_i \quad (n_i \leqq \tau \leqq n_{i+1}),$$

so that

$$\left|1-\frac{r}{a_n}\right|=\left|\frac{n-\tau}{n-\dfrac{\beta_i}{\alpha_i}}\right|=\left|\frac{n-\tau}{n-n_i+\eta_i}\right| \quad (n_i \leqq n \leqq n_{i+1}).$$

Since $\frac{r}{a_n}<1$ for $n>n_{i+1}$, we have by putting $m=[\tau]$, if $\tau \leqq n_{i+1}-1$

$$m(r) \leqq \prod_{n<n_i}\left|\frac{a_n-r}{a_n}\right| \frac{\prod_{n_i}^{m}(\tau-n) \prod_{m+1}^{n_{i+1}}(n-\tau)}{\prod_{n_i}^{n_{i+1}}(n-n_i+\eta_i)}$$

$$\leqq n_{i+1}^{kn_i} \frac{\Gamma(\tau-n_i+1)\Gamma(n_{i+1}-\tau+1)\Gamma(\eta_i)}{\Gamma(\tau-m)\Gamma(m+1-\tau)\Gamma(n_{i+1}-n_i+1+\eta_i)} .$$

Since $\Gamma(z)$ has a pole of the first order at $z=0$, we have from (22),

$$\Gamma(\eta_i) \leqq \text{const.} \frac{1}{\eta_i} \leqq \text{const.}\, n_{i+1}^{k-1} ,$$

so that

$$m(r) \leqq \text{const.}\, n_{i+1}^{k'n_i} \frac{\Gamma(\tau-n_i+1)\Gamma(n_{i+1}-\tau+1)}{\Gamma(n_{i+1}-n_i+1+\eta_i)}$$

$$\leqq \text{const.}\, n_{i+1}^{k'n_i} \frac{\Gamma(\tau-n_i+1)\Gamma(n_{i+1}-\tau+1)}{\Gamma(n_{i+1}-n_i+1)} \quad (k'=k+1) .$$

If $n_i \leqq \tau \leqq n_i+1$, or $n_{i+1}-1 \leqq \tau \leqq n_{i+1}$, then we have easily

$$m(r) \leqq \text{const.}\, n_{i+1}^{k'n_i} . \tag{25}$$

If $n_i+1 \leqq \tau \leqq n_{i+1}-1$, then by Stirling's formula,

$$\Phi(\tau)=\frac{\Gamma(\tau-n_i+1)\Gamma(n_{i+1}-\tau+1)}{\Gamma(n_{i+1}-n_i+1)}$$

$$\leqq \text{const.}\sqrt{\frac{(\tau-n_i)(n_{i+1}-\tau)}{n_{i+1}-n_i}}\left(\frac{\tau-n_i}{e}\right)^{\tau-n_i}\left(\frac{n_{i+1}-\tau}{e}\right)^{n_{i+1}-\tau} \Big/ \left(\frac{n_{i+1}-n_i}{e}\right)^{n_{i+1}-n_i}$$

$$\leqq \text{const.}\sqrt{n_{i+1}}\, \frac{(\tau-n_i)^{\tau-n_i}(n_{i+1}-\tau)^{n_{i+1}-\tau}}{(n_{i+1}-n_i)^{n_{i+1}-n_i}} .$$

Since $(\tau-n_i)^{\tau-n_i}(n_{i+1}-\tau)^{n_{i+1}-\tau}$ attains its maximum at $\tau_0=\frac{n_i+n_{i+1}}{2}$ and its value at τ_0 is $\left(\frac{n_{i+1}-n_i}{2}\right)^{n_{i+1}-n_i}$, we have

$$\Phi(\tau)\leqq\text{const.}\sqrt{n_{i+1}}\,/\,2^{n_{i+1}-n_i}\leqq\text{const.},$$

so that for $n_i^k\leqq r\leqq n_{i+1}^k$,

$$m(r)\leqq\text{const. } n_{i+1}^{k/n_i}, \tag{26}$$

or $$\log m(r)\leqq\text{const. } n_i\log n_{i+1}\leqq\text{const. } n_i^{1+\frac{s}{2}}<n_i^{1+s} =n_i^{\frac{1}{\rho}(1+\delta)(\rho-\varepsilon)}=n_i^{k(1+\delta)(\rho-\varepsilon)}.$$

Hence $$\log m(r)\leqq r^{\rho-\varepsilon}\quad\text{for}\quad n_i^{k(1+\delta)}\leqq r\leqq n_{i+1}^k, \tag{27}$$

so that $$E=E(\log m(r)>r^{\rho-\varepsilon})$$

is contained in $\{I_i\}$, where $I_i=[n_i^k,\ n_i^{k(1+\delta)}]$. Now

$$\sum_{\nu=1}\int_{I_\nu}\frac{dr}{r}=k\delta(\log n_i+\log n_{i-1}+\cdots+\log n_1)\leqq k\delta(\log n_i+(i-1)\log n_{i-1}).$$

Since $n_i\geqq n_{i-1}^2$, we have $i=O(\log\log n_i)$, so that

$$\sum_{\nu=1}^{i}\int_{I_\nu}\frac{dr}{r}\leqq k\delta(\log n_i+O(\log n_{i-1})^2)\leqq k\delta(\log n_i+O(\log\log n_i)^2)$$
$$\leqq k\delta(1+\eta)\log n_i,\quad(\eta\to 0\text{ with } i\to\infty).$$

Hence if $n_i^k\leqq r\leqq n_i^{k(1+\delta)}$,

$$\frac{1}{\log r}\int_{E(1,r)}\frac{dr}{r}\leqq\frac{1}{\log r}\sum_{\nu=1}^{i-1}\int_{I_\nu}\frac{dr}{r}+\frac{1}{\log r}\int_{n_i^k}^{r}\frac{dr}{r}\leqq\frac{k\delta(1+\eta)\log n_{i-1}}{\log r}$$
$$+\left(1-\frac{\log n_i^k}{\log r}\right)\leqq\frac{k\delta(1+\eta)\log n_{i-1}}{k\log n_i}+\left(1-\frac{\log n_i^k}{\log n_i^{k(1+\delta)}}\right)=o(1)+\frac{\delta}{1+\delta}.$$

From this we have

$$\overline{\lambda}(E)=\varlimsup_{r\to\infty}\frac{1}{\log r}\int_{E(1,r)}\frac{dr}{r}\leqq\frac{\delta}{1+\delta}<1-2\rho.$$

5. Some remarks.

1. Let $f(z)$ be an integral function of finite order ρ and D be a domain, which contains $z=0$ and $z=\infty$ lies on its boundary Λ and $\log|f(z)|\leqq k\log r$ ($|z|=r$, $k>0$) on Λ. We define $\overline{\theta}(r)$ for D as in § 3. Let C: $|z|=a$ be a circle contained in D and we choose a constant $K>0$, such that $\log|f(z)|-k\log|z|-K<0$ on C and $\log M(a)+k\log a+K>0$. Let z_0 be a point of D, such that $\log|f(z_0)|-k\log|z_0|-K>0$ ($|z_0|=r_0>a$). Since $\log M(r)$ is a convex function of $\log r$ and $\lim\limits_{r\to\infty}\frac{\log M(r)}{\log r}=\infty$, $\log M(r)-k\log r>0$ for large $r>0$, hence by (6),

$$0<\log|f(z_0)|-k\log|z_0|-K\leqq(\log M(r)-k\log r+K)u_r(z_0)$$
$$\leqq\text{const.}(\log M(r)-k\log r+K)e^{-\pi\int_{2r_0}^{\frac{r}{2}}\frac{dr}{r\theta(r)}}\qquad(r>4r_0).$$

From this we have

Theorem 2. *Let $f(z)$ be an integral function of finite order ρ and $\log|f(z)|\leqq k\log r$ $(k>0)$ on the boundary Λ of an infinite domain D, then*

$$\bar{\lambda}(\Lambda)=\varlimsup_{r\to\infty}\frac{2\pi}{\log r}\int_1^r\frac{dr}{r\bar{\theta}(r)}\leqq 2\rho\ .$$

Theorem 3. *Let $f(z)$ be an integral function of order $\rho(0<\rho<\frac{1}{2})$, then*

$$\underline{\lambda}[E(\log m(r)>k\log r)]\geqq 1-2\rho>0\quad (k>0)\ ^{5)}.$$

Let $\varphi(r)$ be an increasing function of r, such that $\varlimsup_{r\to\infty}\frac{\varphi(r)}{\log r}=\infty$, then for any $0<\rho<1$, there exists an integral function of order ρ, such that

$$\underline{\lambda}[E(\log m(r)>\varphi(r))]=0\ .$$

Proof. The first part follows from Theorem 2, since the set E of r, such that $|z|=r$ meets Λ coincides with $E=E(\log m(r)\leqq k\log r)$ and $\bar{\theta}(r)\leqq 2\pi$ for $r\in E$ and $\bar{\theta}(r)=\infty$ otherwise. We shall prove the second part. We put $k=\frac{1}{\rho}>1$. Since $\varlimsup_{r\to\infty}\frac{\varphi(r)}{\log r}=\infty$, we can choose positive integers n_i, such that $1<n_1<n_2<\dots<n_i\to\infty$, $\frac{n_{i+1}}{n_i}\to\infty$ and

$$\frac{\varphi(n_{i+1}^{\frac{k}{i+1}})}{\log(n_{i+1}^{\frac{k}{i+1}})}\geqq(i+1)^2 n_i\ ,$$

or

$$\varphi(n_{i+1}^{\frac{k}{i+1}})\geqq k(i+1)n_i\log n_{i+1}\ . \tag{28}$$

With these n_i, we construct an integral function $f(z)$ of order ρ as (23) in the proof of Theorem 1 (iii). Then for $n_i^k\leqq r\leqq n_{i+1}^k$, we have by (26), (28),

$$\log m(r)\leqq\text{const. } n_i\log n_{i+1}\leqq\varphi(n_{i+1}^{\frac{k}{i+1}})\leqq\varphi(n_{i+1}^{k\delta})\qquad(0<\delta<1)\ ,$$

so that $\qquad \log m(r)\leqq\varphi(r)\quad\text{for}\quad n_{i+1}^{k\delta}\leqq r\leqq n_{i+1}^k\ .$

Since

$$\int_{I_i}\frac{dr}{r}=(1-\delta)\log n_{i+1}^k\ ,\qquad I_i=[n_{i+1}^{k\delta}\ ,\ n_{i+1}^k]\ ,$$

and δ is arbitrary, we have $\bar{\lambda}[E(\log m(r)\leqq\varphi(r))]=1$, so that

$$\underline{\lambda}[E(\log m(r)>\varphi(r))]=0\ .$$

6. Dirichlet's problem with an unbounded boundary value.

1. Let D be a domain on the z-plane, which contains $z=\infty$ on its boundary Λ and $\varphi(z)$ be a given continuous function on Λ. In the usual Dirichlet's problem, $\varphi(z)$ is assumed to be bounded. If $\varphi(z)$ is unbounded, there exists, in general, no harmonic function in

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D, which assumes the value $\varphi(z)$ on Λ. We consider a special case, where $\varphi(z)=r^k$ ($|z|=r$, $k>0$) and shall prove

Theorem 4. (i) *If*

$$\underline{\lambda}(\Lambda)=\varliminf_{r\to\infty}\frac{2\pi}{\log r}\int_1^r\frac{dr}{r\bar{\theta}(r)}>2k,\ (\alpha=\tfrac{1}{2}\underline{\lambda}(\Lambda)-k>0)\,,$$

then there exists a harmonic function $u(z)$ in D, which assumes the value r^k at regular points of Λ and

$$r^k\leq u(z)\leq \text{const.}\,\frac{1}{r^{\alpha-\varepsilon}}e^{\pi\int_1^{2r}\frac{dr}{r\bar{\theta}(r)}}\qquad (|z|=r)\ in\ D$$

for any $\varepsilon>0$.

(ii) *If* $$\underline{\lambda}^*(\Lambda)=\varliminf_{r/a\to\infty}\frac{2\pi}{\log(r/a)}\int_a^r\frac{dr}{r\bar{\theta}(r)}>2k\,,$$

then $$r^k\leq u(z)\leq \text{const.}\,r^k\qquad in\ D\,.$$

Proof. (i) Let D_r^0, $\bar{\theta}(r)$, $u_r(z)$ be defined as in §3. Then by (6),

$$u_r(z_0)\leq \text{const.}\,e^{-\pi\int_{2r_0}^{\frac{r}{2}}\frac{dr}{r\bar{\theta}(r)}}\qquad (|z_0|=r_0,\ r\geq 4r_0)\,. \tag{29}$$

By the hypothesis,

$$\pi\int_1^r\frac{dr}{r\bar{\theta}(r)}\geq k_1\log r\qquad (k_1>k,\ r\geq R_0)\,, \tag{30}$$

so that

$$u_r(z_0)\leq \text{const.}\,e^{\pi\int_1^{2r}\frac{dr}{r\bar{\theta}(r)}}e^{-\pi\int_1^{\frac{r}{2}}\frac{dr}{r\bar{\theta}(r)}}\leq \text{const.}\,\frac{1}{r^{k_1}}e^{\pi\int_1^{2r_0}\frac{dr}{r\bar{\theta}(r)}}\ (r\geq 4r_0). \tag{31}$$

Let Λ_r be the part of Λ, which lies in $|z|<r$ and $v_r(z)$ be a harmonic function in D, such that $v_r(z)=0$ on Λ_r and $v_r(z)=1$ on $\Lambda-\Lambda_r$ at its regular points. Then

$$v_r(z)\leq u_r(z)\qquad \text{in } |z|<r\,, \tag{32}$$

so that by (31) the integral

$$u(z)=k\int_0^\infty v_r(z)r^{k-1}dr \tag{33}$$

converges and represents a harmonic function in D. We can prove similarly as the proof of Lemma 4, that $u(z)=r^k$ on Λ at its regular points. Hence a harmonic function $u(z)$, which satisfies the condition of the theorem exists.

(ii). Let z_0 ($|z_0|=r_0$) be any point of D. Then by (31), (32), (33),

$$\begin{aligned}u(z_0)&=k\int_0^{4r_0}v_r(z_0)r^{k-1}dr+k\int_{4r_0}^\infty v_r(z_0)r^{k-1}dr\\ &\leq k\int_0^{4r_0}r^{k-1}dr+\text{const.}\,e^{\pi\int_1^{2r_0}\frac{dr}{r\bar{\theta}(r)}}\int_{4r_0}^\infty\frac{dr}{r^{1+k_1-k}}\\ &\leq(4r_0)^k+\text{const.}\,\frac{1}{r_0^{k_1-k}}e^{\pi\int_1^{2r_0}\frac{dr}{r\bar{\theta}(r)}}\,,\end{aligned}$$

Since by (30)

$$e^{\pi\int_1^{2r_0}\frac{dr}{r\bar{\theta}(r)}}\geqq(2r_0)^{k_1} \qquad (2r_0\geqq R_0)\ ,$$

we have

$$u(z_0)\leqq\text{const.}\,\frac{1}{r_0^{k_1-k}}e^{\pi\int_1^{2r_0}\frac{dr}{r\bar{\theta}(r)}}\ .$$

Since k_1 is any number, such that $\frac{1}{2}\underline{\lambda}(\Lambda)>k_1>k$, we have

$$u(z_0)\leqq\text{const.}\,\frac{1}{r_0^{\alpha-\varepsilon}}e^{\pi\int_1^{2r_0}\frac{dr}{r\bar{\theta}(r)}}$$

for any $\varepsilon>0$.

(iii). Next we shall prove that $r^k\leqq u(z)$ in D. Let $V_R(z)$ be a harmonic function in D_R^0, such that $V_R(z)=r^k$ ($|z|=r$) on the whole boundary of D_R^0. Then since r^k is subharmonic, we have

$$r^k\leqq V_R(z) \quad \text{in } D_R^0\ .$$

Let $u(z)$ be the harmonic function constructed in (i), we have by the maximum principle,

$$r^k\leqq V_R(z)\leqq R^k u_R(z)+u(z) \quad \text{in } D_R^0 \quad (R\geqq r)\ .$$

Since by (31), $R^k u_R(z)\to 0$ ($R\to\infty$), we have $r^k\leqq u(z)$ in D.

(iv). If

$$\underline{\lambda}^*(\Lambda)=\varliminf_{r/a\to\infty}\frac{2\pi}{\log(r/a)}\int_a^r\frac{dr}{r\bar{\theta}(r)}>2k\ ,$$

then we can prove similarly as Lemma 4,

$$u(z)\leqq\text{const.}\,r^k \qquad \text{in } D.$$

Hence our theorem is proved.

2. By means of the above theorem, we can prove similarly as Theorem 1 the following theorem.

Theorem 5. *Let $f(z)$ be an integral function of finite order $\rho>0$ and $\log|f(z)|\leqq r^{\rho-\varepsilon}$ ($\varepsilon>0$) on the boundary Λ of an infinite domain D, then*

$$\underline{\lambda}^*(\Lambda)=\varliminf_{r/a\to\infty}\frac{2\pi}{\log(r/a)}\int_a^r\frac{dr}{r\bar{\theta}(r)}\leqq 2\rho\ .$$

Compare this theorem with Theorem 2.

If $f(z)$ is of regular growth, such that

$$\lim_{r\to\infty}\frac{\log\log M(r)}{\log r}=\rho\ ,$$

then the set $\log|f(z)|>r^{\rho-\varepsilon}$ contains an infinite domain for any $\varepsilon>0$. As an application of Theorem 5, we shall prove the following two theorems.

Tehorem 6. *Let $f(z)$ be an integral function of finite order $\rho>0$ and Λ be the closed set of points, such that* $\log|f(z)|\leqq r^{\rho-\varepsilon}$ ($|z|=r$, $\varepsilon>0$)

and Λ_θ be the intersection of Λ with a half-line: arg $z=\theta$. Then

$$\underline{\lambda}^*(\Lambda_\theta)\leqq 2\rho .$$

Proof. Since $\log|f(z)|\leqq r^{\rho-\varepsilon}$ on Λ_θ, if we apply Theorem 5 to the outside of Λ_θ, then we have our theorem, since $\bar{\theta}(r)=2\pi$, when $|z|=r$ meets Λ_θ and $\bar{\theta}(r)=\infty$ otherwise.

Theorem 7. *Let $f(z)$ be an integral function of finite order $\rho>0$ and $M(r)=\underset{|z|=r}{\mathrm{Max.}}|f(z)|$. Then*

$$\bar{\lambda}^*[E(\log M(r)>r^{\rho-\varepsilon})]=1$$

for any $\varepsilon>0$ and for any $0<\rho<\infty$, there exists an integral function of order ρ, such that

$$\bar{\lambda}[E(\log M(r)>r^{\rho-\varepsilon})]<1\,,\quad 0<\varepsilon<\mathrm{Min.}\left(\frac{\rho^2}{1+\rho}\,,\ \frac{\rho}{2}\right).$$

Proof. (i). Let

$$E=E(\log M(r)\leqq r^{\rho_1})\qquad (\rho_1=\rho-\varepsilon)\,, \tag{34}$$

then E consists of a countable number of disjoint closed intervals $I_\nu=[r_\nu, r'_\nu]$ ($\nu=1, 2, \dots$) and

$$\log|f(z)|\leqq r^{\rho_1}\qquad (|z|=r)$$

in the closed ring domain $\Delta_\nu: r_\nu\leqq|z|\leqq r'_\nu$.

We construct a canal in Δ_ν, such that we take off from Δ_ν its part: $|\arg z|<\delta$, $r_\nu\leqq|z|\leqq r'_\nu$ and Δ^0_ν be the remaining closed domain and put $\Delta=\sum_{\nu=1}^{\infty}\Delta^0_\nu$ and let D be the complementary set of Δ. Then D is a connected infinite domain and $\log|f(z)|\leqq r^{\rho_1}$ on its boundary Λ. Hence by Theorem 5

$$\varliminf_{r/a\to\infty}\frac{2\pi}{\log(r/a)}\int_a^r\frac{dr}{r\bar{\theta}(r)}\leqq 2\rho .$$

Since $\bar{\theta}(r)=2\delta$ for $r\in E$ and $\bar{\theta}(r)=\infty$ otherwise, we have

$$\underline{\lambda}^*(E)=\varliminf_{r/a\to\infty}\frac{1}{\log(r/a)}\int_{E(a,\,r)}\frac{dr}{r}\leqq\frac{\rho}{\pi}2\delta\,,$$

so that for $\delta\to 0$, we have $\underline{\lambda}^*(E)=0$, hence

$$\bar{\lambda}^*[E(\log M(r)>r^{\rho-\varepsilon})]=1\,.$$

(ii). Next we shall prove that for any $0<\rho<\infty$, there exists an integral function of order ρ, such that

$$\bar{\lambda}[E(\log M(r)>r^{\rho-\varepsilon})]<1\,,\quad 0<\varepsilon<\mathrm{Min.}\left(\frac{\rho^2}{1+\rho}\,,\ \frac{\rho}{2}\right).$$

First suppose that $0<\rho<1$ and $0<\varepsilon<\dfrac{\rho^2}{1+\rho}$, then

$$\rho-\varepsilon>\frac{\rho}{1+\rho}>0\,,\quad \frac{1}{\rho}(1+\rho)(\rho-\varepsilon)=1+s\quad (s>0).$$

With this $s>0$, we construct an integral function $f(z)$ of order ρ as (23) in the proof of Theorem 1 (iii). Let

$$n_i^{k+1} \leqq r \leqq n_{i+1}^{k-1} \qquad \left(k=\frac{1}{\rho}>1\right), \tag{35}$$

then

$$M(r)=\prod_{n<n_i}\left(1+\frac{r}{a_n}\right)\prod_{n}^{n_{i+1}}\left(1+\frac{r}{a_n}\right)\prod_{n>n_{i+1}}\left(1+\frac{r}{a_n}\right)=\Pi_1\cdot\Pi_2\cdot\Pi_3\,. \tag{36}$$

Now

$$\Pi_1 \leqq (2n_{i+1})^{kn}\,. \tag{37}$$

Since $a_n=\alpha_i n-\beta_i$, $a_{n_i}=n_i^k$, $\alpha_i \sim n_{i+1}^{k-1}$,

$$\begin{aligned}\log \Pi_2 &= \sum_{n_i}^{n_{i+1}}\log\left(1+\frac{r}{a_n}\right) \leqq \log\left(1+\frac{r}{a_{n_i}}\right)+\int_{n_i}^{n_{i+1}}\log\left(1+\frac{r}{\alpha_i x-\beta_i}\right)dx\\ &\leqq \log\left(1+\frac{r}{n_i^k}\right)+\int_{n_i}^{n_{i+1}}\frac{r}{\alpha_i x-\beta_i}dx \leqq k\log n_{i+1}+\frac{r}{\alpha_i}\log\frac{n_{i+1}^k}{n_i^k}\\ &\leqq k\log n_{i+1}+\text{const.}\ \frac{n_{i+1}^{k-1}}{n_{i+1}^{k-1}}\log n_{i+1} \leqq \text{const.}\log n_{i+1}\,.\end{aligned} \tag{38}$$

Similarly for $j \geqq i+1$,

$$\begin{aligned}\sum_{n_j}^{n_{j+1}}\log\left(1+\frac{r}{a_n}\right) &\leqq \log\left(1+\frac{r}{n_j^k}\right)+\frac{r}{\alpha_j}\log\frac{n_{j+1}^k}{n_j^k}\\ &\leqq \text{const.}\ r\left(\frac{1}{n_j^k}+\frac{1}{n_{j+1}^{k-1}}\log n_{j+1}\right) \leqq \text{const.}\ \frac{r}{n_j^k}\,,\end{aligned}$$

so that

$$\log \Pi_3 \leqq \text{const.}\ r\sum_{\nu=1}^{\infty}\frac{1}{n_{i+\nu}^k}\,.$$

Since $n_{i+1} \geqq 2n_i$, $n_{i+\nu} \geqq 2^{\nu-1}n_{i+1}$, we have

$$\begin{aligned}\log \Pi_3 &\leqq \text{const.}\ \frac{r}{n_{i+1}^k}\sum_{\nu=0}^{\infty}\frac{1}{2^{k\nu}} \leqq \text{const.}\ \frac{r}{n_{i+1}^k} \leqq \text{const.}\ \frac{n_{i+1}^{k-1}}{n_{i+1}^k}\\ &= \text{const.}\ \frac{1}{n_{i+1}} \to 0\,.\end{aligned} \tag{39}$$

Hence from (37), (38), (39),

$$\begin{aligned}\log M(r) &\leqq \text{const.}\ n_i\log n_{i+1} \leqq \text{const.}\ n_i^{1+s/2} < n_i^{1+s}\\ &= n_i^{(k+1)(\rho-\varepsilon)} \leqq r^{\rho-\varepsilon} \quad \text{for} \quad n_i^{k+1} \leqq r \leqq n_{i+1}^{k-1},\end{aligned}$$

so that $E=E(\log M(\mathrm{r})>r^{\rho-\varepsilon})$ is contained in $\{I_i\}$, where $I_i=[n_i^{k-1}, n_i^{k+1}]$. Since

$$\int_{I_i}\frac{dr}{r}=\frac{2\rho}{1+\rho}\log n_i^{k+1}\,,$$

we have similarly as the proof of Theorem 1 (iii),

$$\bar{\lambda}(E)=\overline{\lim_{r\to\infty}}\frac{1}{\log r}\int_{E(1,r)}\frac{dr}{r} \leqq \frac{2\rho}{1+\rho}<1\,.$$

Next suppose that $1 \leqq \rho < \infty$. We choose a rational number

$\lambda=p/q$, such that $\lambda>\rho$, where p, q are positive integers. Then $\rho_1=\rho/\lambda<1$. We construct an integral function $f_1(z)$ of order ρ_1, such that

$$\bar{\lambda}[E(\log M_1(r)>r^{\rho_1-\varepsilon_1})]<1\,, \quad 0<\varepsilon_1<\frac{\rho_1^2}{1+\rho_1}=\frac{\rho^2}{\lambda(\lambda+\rho)}\,, \tag{40}$$

where $M_1(r)=\underset{|z|=r}{\mathrm{Max.}}|f_1(z)|$. We put $z=w^\lambda$ and let

$$f(w)=\prod_{\nu=0}^{q-1}(f_1(\omega^\nu z)-a)=\prod_{\nu=0}^{q-1}(f_1(\omega^\nu w^\lambda)-a) \quad (\omega=e^{\frac{2\pi i}{q}})\,,$$

$$M(R)=\underset{|w|=R}{\mathrm{Max.}}|f(w)| \qquad (|w|=R,\ |z|=r,\ r=R^\lambda)\,. \tag{41}$$

Then for a certain a $f(w)$ is an integral function of order ρ[6] and

$$M(R)\leqq[M_1(r)]^{q+1} \quad (r\geqq r_0). \tag{42}$$

Since the logarithmic density is invariant for the transformation $r=R^\lambda$, we have from (40), (42),

$$\bar{\lambda}[E(\log M(R)>(q+1)R^{\lambda\rho_1-\lambda\varepsilon_1})]=\bar{\lambda}[E(\log M(R)>(q+1)R^{\rho-\lambda\varepsilon_1})]<1\,.$$

Hence for any $0<\eta<1$,

$$\bar{\lambda}[E(\log M(R)>R^{\rho-\eta\lambda\varepsilon_1})]<1\,. \tag{43}$$

Since $\eta\lambda\varepsilon_1<\eta\dfrac{\rho^2}{\lambda+\rho}\to\dfrac{\rho}{2}$ for $\eta\to1$, $\lambda\to\rho$, we have

$$\bar{\lambda}[E(\log M(R)>R^{\rho-\varepsilon})]<1$$

for any $\varepsilon<\rho/2$. Since

$$\mathrm{Min.}\left(\frac{\rho^2}{1+\rho}\,,\ \frac{\rho}{2}\right)=\frac{\rho^2}{1+\rho} \quad \text{for } 0<\rho\leqq1\,,$$
$$=\frac{\rho}{2} \qquad \text{for } 1\leqq\rho<\infty\,,$$

our theorem is proved.

6) G. Valiron: Lectures on the general theory of integral functions. p. 190.