26. *On the Behaviour of the Boundary of Riemann Surfaces, II.*

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In Part I of the same title[1] we have dealt with the absolute harmonic measure of the boundary of Riemann surfaces and defined the kind of their connected pieces. In this paper we will investigate chiefly the behaviour of the boundary of Riemann surfaces of the first kind.

§ 1. Some theorems on the boundary of Riemann surfaces of the first kind.

Theorem 1. Let $\bar{F}$ be a connected piece of the first kind with the relative boundary (Γ^i) and the proper boundary E of a Riemann surface F spread over the z-plane and $\mathfrak{F}_\rho$ be any connected piece of $\bar{F}$, whose boundary includes no point of (Γ^i), lying above a disc K: $|z-a|<\rho$, where a is an arbitrary point on the z-planes. Then $\mathfrak{F}_\rho$ covers any point inside of K at least once, except a set of points of capacity zero.[2]

Proof. (i) The boundary of $\mathfrak{F}_\rho$ is both E and Jordan curves (γ), whose projections on the z-plane coincide with the circumference γ: $|z-a|=\rho$.

At this time we will prove that there exists on $\mathfrak{F}_\rho$ no non-constant bounded harmonic function $v(z)$, which has the next condition:

$$v(z)=0 \text{ on } (\gamma), \qquad 0 \leqq v(z) \leqq 1 \text{ on } \mathfrak{F}_\rho .$$

Suppose that $v(z_0)>0$, $z_0 \in \mathfrak{F}_\rho$.

We consider approximating function $\bar{u}_n(z)$ and let $z_0 \in \bar{F}_n$.

At this time let $\mathfrak{F}_\rho^{(n)}$ one of the cross-cuts $\mathfrak{F}_\rho \cdot \bar{F}_n$, which includes z_0.

Then the boundary of $\mathfrak{F}_\rho^{(n)}$ is both (γ) and (C_n^j).

Since from above

$$\begin{aligned} &v(z)=0 \text{ on } (\gamma), && v(z) \leqq 1 \text{ on } (C_n^j), \\ &\bar{u}_n(z) \geqq 0 \text{ on } (\gamma), && \bar{u}_n(z)=1 \text{ on } (C_n^j), \end{aligned}$$

then by the maximum principle

$$v(z) \leqq \bar{u}_n(z) \text{ on } \mathfrak{F}_\rho^{(n)} \text{ and } v(z_0) \leqq \bar{u}_n(z_0) .$$

Since for $n \to \infty$ $\bar{u}_n(z) \to \bar{u}(z) \equiv 0$, $v(z_0)$ must be zero, which is absurd, q.e.d.

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1) Y. Nagai: On the Behaviour of the Boundary of Riemann Surfaces, I. Proc. Jap. Acad., vol. 26 (1950).

2) In this paper, "capacity" means the logarithmic capacity.

(ii) Suppose that $\mathfrak{F}_p$ does not cover a set M of positive capacity lying inside of K. Let M_0 be the subset of M, which lies inside of a Jordan curve γ_0, where γ_0 lies inside of K and has a positive distance from γ. Then M_0 is a closed set. If γ_0 lies near to γ, then cap. $M_0 > 0$.

Let D be the schlicht domain bounded by γ and M_0 and $v(z)$ be the solution of Dirichret problem with the boundary condition:

$$v(z) = 1 \text{ on } M_0, \qquad v(z) = 0 \text{ on } \gamma.$$

Then since cap. $M_0 > 0$, $v(z) \not\equiv 0$.

When we consider $v(z)$ as a bounded harmonic function on $\mathfrak{F}_p$, then $v(z)$ has the property, which we have mentioned at (i) and therefore $v(z) \equiv 0$, which is absurd.

Hence our assertion has been proved, q.e.d.

Definition. We call the number of sheets of a Riemann surface F above z, the number of covering time of z by F and denote it by $n(z)$. Then we shall prove;

Theorem 2. Let F be a Riemann surface spread over the z-plane and have no Green's function. If $n(a) \geqq k$, then the set of such z-points as $n(z) \leqq k-1$ is of cap. 0.

Proof. We can easily know that the set of such points z that $n(z) \geqq k$ makes a domain D. Let the boundary of D be E, then obviously $n(z) \leqq k-1$ for any point z belonging to E. Therefore we will prove that cap. of $E = 0$.

Now we suppose that cap. of $E > 0$. Let the set of such z-points belonging to E as $n(z) = m$, be e_m, then $E = \sum_{m=0}^{k-1} e_m$.

By the assumption that cap. of $E > 0$, there is a certain e_m, such that cap. of $e_m > 0$, $(m \leqq k-1)$. As well-known we can find a closed subset E_m of e_m, such that cap. of $E_m > 0$.

Then there exists a point z_0 on E, such that cap. of $E_m \cdot K > 0$ for any small disc K about z_0, where $E_m \cdot K$ is the part of E contained in K. Since $z_0 \in E_m$, z_0 is covered m-times by F. Hence the part of F above K contains m discs: $F_1, F_2, \ldots, F_m$ consisting of only inner points of F. Since $m \leqq k-1$, there is another connected piece F_0 of F above K other than $F_1, F_2, \ldots\ldots, F_m$.

Then F_0 does not cover $E_m \cdot K$. On the other hand F has no Green's function, so that by Theorem 2, Part I and Theorem 1 we can know that cap. of $E_m \cdot K$ and therefore cap. of E must be zero, q.e.d.

From this theorem we have immediately next colloraries.

Collorary 1. Let F be a Riemann surface spread over the z-plane and have no Green's function. If the set of such z-points as $n(z) = k$ is cap. positive, then the whole points on the z-plane are covered exactly k-times by F except a possible set of z-points of cap. 0, $n(z)$ of which is $\leqq k-1$.

Collorary 2. Let F be a Riemann surface spread over the z-plane and have no Green's function. If there exists such a point a that $n(a)=\infty$, then the whole points on the z-plane are covered infinitely often by F except a possible set of z-points of cap. 0.

Specially for Riemann surfaces consisting of a finite number of sheets;

Theorem 3. Let F be a Riemann surface consisting of a finite number of sheets spread over the z-plane and have no Green's function. Then the set E of the projections on the z-plane of the boundary of F is of cap. 0.

Proof. If F consists of m-sheets, then there exists such a point a as $n(a)=m$. Then by Theorem 2 the set e of such z-points as $n(z)\leqq m-1$ is of cap. 0. On the other hand any point of E is ever contained in e. Thus cap. of E is zero, q.e.d.

Theorem 4. (Tsuji's theorem)[3] Let F be a Riemann surface consisting of a finite number of sheets spread over the z-plane and the set E of the projections on the z-plane of its boundary be of cap. 0. Then F has no Green's function.

Proof. Suppose that F has Green's function. We map the universal covering surface $\mathfrak{F}^\infty$ of F on $|w|<1$ by $z=f(w)$. Then obviously F satisfies the necessary conditions of Theorem 1 of Part I, so that by that theorem $\lim f(w)$ exists almost everywhere on $|w|=1$, when w tends to $|w|=1$ non-tangentially. Obviously this limiting values belong to E and cap. of $E=0$, so that by R. Nevanlinna's theorem[4] $f(w)\equiv$ const., which is absurd.

Hence F has no Green's function, q.e.d.

Next we enunciate similar theorems about partial Riemann surface $\bar{F}$ of F, which has the relative boundary (Γ^i) and the proper boundary E.

Now let (D_j) be a finite number of domains on the z-plane, which are separated by the projections on the z-plane of (Γ^i).

Then we shall prove;

Theorem 2′. Let $\bar{F}$ be of the first kind. If $n(a)\geqq k$ for $a\in D_j$, then the set of such z-points in D_j as $n(z)\leqq k-1$ is of cap. 0.

Proof. Since $\bar{F}$ is of the first kind, considering Theorem 1 we can prove the assertion quite similarly as Theorem 2. Thus we have

Collorary 1′. Let $\bar{F}$ be of the first kind. If the set of such z-points belonging to a certain D_j as $n(z)=k_j$ is of cap. positive, then the whole points of D_j are covered exactly k_j-times by $\bar{F}$ with

3) M. Tsuji: Theory of conformal mapping of multiply connected domain, III. Jap. Journ. Math., **19** (1944), p. 170.

4) R. Nevanlinna: Eindeutige analytische Funktionen, (1936), p. 198.

a possible exception of cap. 0. Moreover at this time for the other D_j' too we can find the same finite number k_j' as k_j.

Proof. The first part of the assertion can be proved by Theorem 2'. Next let $z_0 \in D_j$ be such a point as $n(z_0) = k_j$ lying suitably near the boundary of D_j and D_j' be one of the adjoining domains to D_j. By a suitable slight deformation of (Γ^i) we can consider z_0 as an inner point of D_j'. At this time the change of $n(z_0)$ is obviously finite such that

$$\text{the new} \quad n(z_0) = k_j + m_j' = k_j' \ (< \infty),$$

and a suitable neighbourhood of z_0 is covered exactly k_j'-times by $\bar{F}$. As above stated the nature of $\bar{u}(z) \equiv 0$ is independent of a slight deformation of (Γ^i), so that Theorem 1 is reserved for D_j'.

Therefore applying the first part of the assertion on D_j', we can know that the whole point of the former D_j' are exactly k_j'-times covered by $\bar{F}$ with a possible exception of cap. 0. Continuing such a way we can know the validity of our assertion, q.e.d.

Collorary 2'. Let $\bar{F}$ be of the first kind and $n(a) = \infty$, then for the whole z on the z-plane $n(z) = \infty$ except a possible set of z-points of cap. 0.

Proof. Let a belong to D_j, then quite similarly as above stated we can prove that every point of D_j is covered by $\bar{F}$ infinitely often except a possible set of z-points of cap. 0. Then by a suitable slight deformation of (Γ^i) we can claim the same property for each of the adjoining domains D_j' of D_j and therefore for all (D_j). Consequently the assertion is true, q.e.d.

Theorem 3'. Let $\bar{F}$ be of the first kind and consist of a finite number of sheets. Then the set e of the projections on the z-plane of the proper boundary E is of cap. 0.

Proof. In the first place we will prove that the cap. of $e \cdot D_j$ for every j is zero, where $e \cdot D_j$ is the part of e lying in D_j. Since $\bar{F}$ consists of a finite number of sheets, the maximum of $n(z)$, when z varies in D_j, is finite k_j. Then as in Collorary 1' stated $n(z) = k_j$ for the whole z-points of D_j with a possible exception ε_j of cap. 0. Of course any point of $e \cdot D_j$ belongs to ε_j, so that $e \cdot D_j$ is of cap. 0, q.e.d.

Next for the part e_Γ of e, which lies on the boundary of (D_j), by a suitable slight deformation of (Γ^i) we can prove quite similarly as above that e_Γ is of cap. 0, q.e.d.

Theorem 4'. Let $\bar{F}$ consist of a finite number of sheets with the relative boundary (Γ^i) and the set ε of the projections on the z-plane of the proper boundary E be of cap. 0. Then $\bar{F}$ is of the first kind.

Proof. Suppose that $\bar{F}$ is of the second kind and therefore its harmonic measuring function $\bar{u}(z) \neq 0$.

Since $\bar{F}$ has Jordan closed curves (Γ^i) as its boundary, it has of course Green's function. Then when we map the universal covering surface $\mathfrak{F}^\infty$ of $\bar{F}$ on $|w|<1$, (Γ^i) and E correspond to (γ) and e on $|w|=1$ respectively, where (γ) is the set of arcs and by Theorem 1, Part I $m(\gamma)+m\,e=2\pi$.

Since cap. of ε is 0, by R. Nevanlinna's theorem we have $m\,e=0$ and $m(\gamma)=2\pi$.

Now we consider the transform $U(w)$ of $\bar{u}(z)$, then since $\bar{u}(z)=0$ on (Γ^i), $U(w)$ must be zero on (γ). While $m(\gamma)=2\pi$, so $U(w)\equiv 0$. Consequently $\bar{u}(z)\equiv 0$, which is absurd.

Hence $\bar{F}$ must be of the first kind, q.e.d.

Remark. Let $\bar{F}$ be of the first kind. The number of the connected pieces of $\bar{F}$ lying above a disc K about any paint a on the z-plane is in general enumerably infinite. Let $\bar{F}_0$ be one of them and e_0 be its proper boundary, the set ε_0 of whose projections on the z-plane lies inside of K. If $\bar{F}_0$ consists of a finite number of sheets, then quite similarly as Theorem 3′ we can prove that ε_0 is of cap. 0. If $\bar{F}_0$ consists of enumerably infinite number of sheets, then similarly as Collorary 2′ we can prove that the whole points of K are covered infinitely often by F_0 with a possible exception of cap. 0.

§ 2. Applications on the transcendental singularities of Riemann surfaces associated with analytic functions.

Definition. Let F be a Riemann surface with the boundary E spread over the z-plane and consist of a finite number of sheets. Then we can ever find such a point z_0 on the z-plane that any point of the set of the projections of E does not coincide with z_0 and therefore all $z_k\,(k=1,2,\ldots,m)$ are inner poins of F, where the projections of all z_k coincide with z_0.

We call for convenience sake such point z_0 a "super-ordinary point of order m" and each of z_k a point "belonging to z_0". Next moreover let F be of the first kind. Then by Theorem 3′ the set e of the projections of the proper boundary E is cap. 0. Now let (a) be a boundary point with the projection a. Then we can draw a simple closed curve Γ on the z-plane surrounding a, on which no point of e lies and whose diameter is chosen to be as small as possible.

At this time we assume that any point belonging to the set of the projections of branch points of F does not lie on Γ. Then the connected piece $\bar{F}_0$ of F with the relative boundary (Γ_0^i) and with (a) as its proper boundary point lying above the domain $\bar{\varDelta}_\Gamma$, where $\bar{\varDelta}_\Gamma$ is the Jordan domain bounded by Γ, is of the first kind and the projections of (Γ_0^i) coincide with Γ. We call this connected piece $\bar{F}_0$ "the Γ-neighbourhood of (a)".

Theorem 5. Let F and $\mathfrak{F}$ be Riemann surfaces spread over the z-plane and the w-plane respectively and both correspond in a one-one conformal manner each other by means of $w=f(z)$, $z=\varphi(w)$.

Let $\bar{F}$ be a part of F, of the first kind, consisting of a finite number of sheets with the relative boundary (Γ^i) and the proper boundary E, the set of whose projections is e, and $\bar{\mathfrak{F}}$ be its transform by $w=f(z)$ spread over the w-plane. If $\bar{\mathfrak{F}}$ consists of a finite number of sheets, then $w=f(z)$ has no transcendental singularity on E.

Proof. Let (a) be an arbitrary point of E, whose projection is a. By the assumption of the finiteness of the number of sheets of $\bar{\mathfrak{F}}$ we can find a "super-ordinary point of order m" w_0, the inner points "belonging to" which are w_k $(k=1, 2, \ldots, m)$, and let z_k $(k=1, 2, \ldots, m)$ be the transform on $\bar{F}$ of those m points by means of $z=\varphi(w)$. Next we take a Γ-neighbourhood of (a), $\bar{F}_0$, which excludes all z_k and consists of n-sheets.

Let $w_i(z)$ $(i=1, 2, \ldots, n)$ be the branch functions of $w=f(z)$ on $\bar{F}_0$. Then since $\dfrac{1}{f(z)-w_0}$ is bounded on $\bar{F}_0$, $\dfrac{1}{w_i(z)-w_0}$ are uniformly bounded in the domain Δ_Γ, where Δ_Γ is the domain bounded by Γ and e. We put

$$\prod_{i=1}^{n}\left(\frac{1}{w-w_0}-\frac{1}{w_i(z)-w_0}\right)=\frac{1}{(w-w_0)^n}+\frac{a_1(z)}{(w-w_0)^{n-1}}+\ldots+a_n(z).$$

Then all $a_i(z)$ are uniform meromorphic functions in Δ_Γ and from above are bounded. Hence by R. Nevanlinna's theorem[5] they are regular at any point belonging to e in the interior of Γ.

Namely $w=f(z)$ has no transcendental singularity at (a).

Since (a) is arbitrary, our assetion has been proved, q.e.d.

From this theorem we can prove immediately;

Theorem 6. Under the same conditions as Theorem 5 if $w=f(z)$ has at least one essential singularity on E, then $w=f(z)$ takes every value infinitely often except a possible set of w-values of cap. 0.

Proof. The transform $\bar{\mathfrak{F}}$ must consist of enumerably infinite number of sheets by Theorem 5, because $\bar{F}$ has at least one essential singularity on E. Since $\bar{F}$ is of the first kind, $\bar{\mathfrak{F}}$ is of the first kind by Theorem 4, Part I. Hence by Collerary $2'$ the whole points on the w-plane are covered infinitely often by $\bar{\mathfrak{F}}$ with a possible exception of cap. 0.

In other words $w=f(z)$ takes every value infinitely often except a possible set of w-values of cap. 0, q.e.d.

5) R. Nevanlinna: loc. cit. 4), p. 132.

This is equivalent to next Noshiro's theorem on generalized algebroidal functions.

Kametani-Tsuji-Noshiro's theorem.[6] Let E be a bounded closed set, of cap. 0, contained entirely inside a domain D and be k-valued algebroidal in the domain D–E such that $w = w(z)$ has at least one essential singularity at some point z_0 belonging to E. Then in any neighbourhood of z_0, $w = w(z)$ takes every value infinitely often except a possible set of w-values of cap. 0.

Proof. Let F be the Riemann surface associated with $w = w(z)$ and (a) be its essential singularity with the projection a belonging to E. Since E is of cap. 0, we can find a Γ-neighbourhood $\bar{F}_0$ of (a), where the diameter of Γ is chosen to be as small as possible. Then since $\bar{F}_0$ satisfies the conditions of Theorem 6, by Theorem 6 $w = w(z)$ takes every value infinitely often with a possible exception of cap. 0 on $\bar{F}_0$. Since the diameter of Γ is arbitrary, the assertion has been proved, q.e.d.

Theorem 7. (Tsuji-Noshiro's theorem)[7] Under the same conditions as Theorem 6, we suppose that the transform $\bar{\mathfrak{F}}$ of $\bar{F}$ by means of $w = f(z)$ has a transcendental singularity (w_0) whose projection w_0. Let Φ_ρ be the ρ-neighbourhood of the accesible boundary point (w_0); namely Φ_ρ is the connected piece of $\bar{\mathfrak{F}}$ with (w_0) as its boundary lying above the disc K: $|w - w_0| < \rho$.

Then Φ_ρ covers every point inside K infinitely often with a possible exception of cap. 0.

Proof. Suppose that Φ_ρ consists of a finite number of sheets.

Then considering a Γ-neighbourhood of (w_0)[8] and the finiteness of the number of sheets of $\bar{F}$, by Theorem 5 we have a contradiction to the assumption that (w_0) is a transcendental singularity.

Hence Φ_ρ must consist of enumerably infinite number of sheets.

Then by the remark in § 1 Φ_ρ covers every point inside K infinitely often with a possible exception of cap. 0, q.e.d.

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6) K. Noshiro: Contribution to the theory of the singularities of analytic functions. Jap. Journ. Math. **19** (1948), p. 325.

7) K. Noshiro: loc. cit. 6) p. 326.

8) Cf. the remark in § 1.