39. On Conformal Slit Mapping of Multiply-Connected Domains.

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(Comm. by K. KUNUGI, M.J.A., July 12, 1950.)

1. We choose, as a basic domain of standard type in the theory of conformal mapping of $n\,(\geqq 3)$-ply connected domains, a concentric circular ring cut along $n-2$ disjoint concentric circular slits, and denote the boundary components of such a domain, D_Q say, by

$$C^{(1)}:\ |z|=1;\qquad C^{(2)}:\ |z|=Q\ (<1);$$
$$C^{(j)}:\ |z|=m_j,\quad \theta_j\leqq \arg z\leqq \theta_j'\ (3\leqq j\leqq n).$$

Each domain bounded by $n\,(\geqq 3)$ disjoint continua possesses $3n-6$ (real) conformal invariants as its *moduli*. For instance, the moduli of the circular slit annulus D_Q may be given by the $3n-6$ quantities

$$Q,\ m_j\ (3\leqq j\leqq n),\ \theta_j-\theta_3\ (3<j\leqq n),\ \theta_j'-\theta_3\ (3\leqq j\leqq n).$$

Let D_{Q_0} be a domain on the w-plane conformally equivalent to D_Q and obtained from a circular slit annulus of the same type as D_Q by cutting along a slit (Jordan arc) Γ_{Q_0} which starts from a point on the exterior boundary component $|w|=1$. An extremal property given by Rengel[1] shows that the radius Q_0 of the interior boundary component of D_{Q_0} never exceeds Q and is, moreover, always less than Q provided D_{Q_0} does not coincide with D_Q. Let now the function mapping D_Q schlicht and conformally onto D_{Q_0} in such a way that the both peripheral boundary circumferences correspond each other, be denoted by

$$w=f(z),\qquad f(Q)=Q_0;$$

this mapping function is uniquely determined under the additional condition explicitly written here.

In case of simply-connected domains, the *Löwner's differential equation* for slit mapping has been recognized as a very fruitful instrument in the theory, a brief proof of which may be given by making use of Poisson formula for functions regular analytic in a circle.[2] This equation can also be generalized to the doubly-con-

1) E. Rengel: Über einige Schlitztheoreme der konformen Abbildung. Schriften d. Math. Sem. u. Inst. f. angew. Math. d. Univ. Berlin **1** (1932/3), 141–162. Cf. also H. Grötzsch: Über einige Extremalprobleme der konformen Abbildung, I. Leipziger Berichte **80** (1928), 367–376.

2) Y. Komatu: Über einen Satz von Herrn Löwner. Proc. Imp. Acad. Tokyo **16** (1940), 512–514.

nected case with aid of Villat's formula for functions regular analytic in an annulus.[3),4)] The aim of the present Note is *to generalize the equation to the case of multiply-connected domains. We shall, namely, introduce a one-parameter family of mapping functions:*

$$\{f(z, q)\}, \qquad Q \geqq q \geqq Q_0,$$

connecting the terminal functions z and $f(z)$, and then obtain a differential equation satisfied by $f(z, q)$ as a function of the parameter q.

2. We suppose that a sub-arc of Γ_{Q_0} possessing interior end-point common with it be deleted from the boundary of D_{Q_0}. Let

$$w = h(w_q, q), \qquad h(q, q) = Q_0,$$

be the function which maps an annulus with $n-2$ concentric circular slits onto the domain thus obtained, and hence containing D_{Q_0}, in such a manner that the interior and exterior boundary circumferences correspond each other, q denoting here the radius of the interior boundary circunference of the equivalent domain laid on w_q-plane. The domain corresponding to D_{Q_0} itself by this mapping be D_q, namely we put $h(D_q, q) = D_{Q_0}$. Then D_q is a domain of the same type as D_{Q_0} whose slit starting from a point on $|w_q| = 1$ is an arc corresponding to the deleted part of Γ_{Q_0} by the mapping $w = h(w_q, q)$. By monotony character of the modulus q, the points on Γ_{Q_0} and the values of the parameter q $(Q \geqq q \geqq Q_0)$ correspond in a one-to-one and monotonic way. Let the function mapping D_Q onto D_q be denoted by

$$w_q = f(z, q), \qquad f(Q, q) = q.$$

The condition added here together with the one that the interior and exterior circumferences should correspond each other respectively determines the mapping function uniquely. Obviously, the functional equation

$$f(z) \equiv f(z, Q_0) = h(f(z, q), q)$$

holds good, and also $f(z, Q) = z$. The boundary components of the circular slit anulus[5)] $D_q + \Gamma_q$ be denoted by

3) Y. Komatu: Untersuchungen über konforme Abbildung von zweifach zusammenhängenden Gebieten. Proc. Phys.-Math. Soc. Japan **25** (1943), 1-42.

4) A brief proof of the Villat's formula is given in Y. Komatu, Sur la représentation de Villat pour les fonctions analytiques définies dans un anneau circulaire concentrique. Proc. Imp. Acad. Tokyo **21** (1945), 94 96.

5) $D_q + \Gamma_q$ denote the domain obtained from D_q by deleting from its boundary the slit Γ_q except the peripheral end-point.

$$C_q^{(1)}: \ |w_q| = 1; \qquad C_q^{(2)}: \ |w_q| = q\,(<1);$$

$$C_q^{(j)}: \ |w_q| = m_j(q), \quad \theta_j(q) \leqq \arg w_q \leqq \theta_j'(q) \ (3 \leqq j \leqq n),$$

and the starting point on $C_q^{(1)}$ of the slit Γ_q be $\bar{\gamma}(q) \equiv e^{-i\theta}(q)$.

Now, the function mapping D_q onto D_{q^*} with $Q > q > q^* > Q_0$ is given by

$$W_{q^*} = f(f^{-1}(w_q, q), q^*) \equiv \Phi(w_q; q, q^*) \equiv \Phi(w_q).$$

The points which correspond to the point $w_{q^*} = \bar{\gamma}(q^*)$, being doubly counted as boundary elements, by this mapping be denoted by $e^{i\beta_0(q,q^*)}$ and $e^{i\beta_1(q,q^*)}$ with $\beta_0 < \beta_1$. We then have

$$|\Phi(w_q)| = 1 \qquad (|w_q| = 1, \quad \beta_1 \leqq \arg w_q \leqq \beta_0 + 2\pi),$$

$$|\Phi(w_q)| = q^* \qquad (|w_q| = q, \quad 0 \leqq \arg w_q \leqq 2\pi).$$

We introduce the *Green function* $G(\omega, w_q; q)$ of the circular slit annulus $D_q + \Gamma_q$ laid on ω-plane possessing the pole at w_q, and denote a harmonic function conjugate to $G(\omega, w_q; q)$ with respect to the variable $w_q = u_q + iv_q$ by

$$H(\omega, w_q; q) = \int^{w_q} \left(\frac{\partial G(\omega, w_q; q)}{\partial u_q} dv_q - \frac{\partial G(\omega, w_q; q)}{\partial v_q} du_q \right),$$

and then put

$$F(\omega, w_q; q) = G(\omega, w_q; q) + iH(\omega, w_q; q).$$

Now, $\lg|\Phi(w_q)/w_q|$ is a harmonic function regular and one-valued in $D_q + \Gamma_q$. Hence, denoting by $\partial/\partial\nu$ the differentiation at boundary point ω in the direction of interior normal, we have

$$\lg\left|\frac{\Phi(w_q)}{w_q}\right| = \frac{1}{2\pi}\int \lg\left|\frac{\Phi(\omega)}{\omega}\right| \frac{\partial G(\omega, w_q; q)}{\partial\nu} ds,$$

the integration being taken along the total boundary of $D_q + \Gamma_q$ with length parameter s in the positive sense. The branch which takes the real value $\lg(q^*/q)$ at $w_q = q$ is regular analytic and one-valued in $D_q + \Gamma_q$ and is expressed by the formula

$$\lg\frac{\Phi(w_q)}{w_q} = \frac{1}{2\pi}\int \lg\left|\frac{\Phi(\omega)}{\omega}\right| \frac{\partial F(\omega, w_q; q)}{\partial\nu} ds + ic,$$

c being a real constant. Remembering now the behavior of the function $\Phi(w_q)$ on both boundary circumferences, we have

$$\lg \frac{\Phi(w_q)}{w_q} = \frac{1}{2\pi}\int_{\beta_0(q,q^*)}^{\beta_1(q,q^*)} \lg|\Phi(e^{i\varphi})|\frac{\partial F(e^{i\varphi}, w_q; q)}{\partial \nu}\, d\varphi$$

$$-\frac{1}{2\pi}\int_0^{2\pi} \lg\frac{q^*}{q}\frac{\partial F(qe^{i\varphi}, w_q; q)}{\partial \nu}\, q\, d\varphi$$

$$+\frac{1}{2\pi}\sum_{j=3}^{n}\int_{C_q^{(j)}} \lg\frac{m_j(q^*)}{m_j(q)}\frac{\partial F(\omega, w_q; q)}{\partial \nu}\, ds + ic,$$

each integration in the last sum being taken along the both sides of each circular slit $C_q^{(j)}$ $(3 \leqq j \leqq n)$ in the positive sense with respect to $D_q + \Gamma_q$. Putting $w_q = f(z, q)$, the last equation becomes

$$\lg \frac{f(z,q^*)}{f(z,q)} = \frac{1}{2\pi}\int_{\beta_0(q,q^*)}^{\beta_1(q,q^*)} \lg|\Phi(e^{i\varphi})|\frac{\partial F(e^{i\varphi}, w_q; q)}{\partial \nu}\, d\varphi$$

$$-\frac{1}{2\pi}\lg\frac{q^*}{q}\int_0^{2\pi} \frac{\partial F(qe^{i\varphi}, w_q; q)}{\partial \nu}\, q\, d\varphi$$

$$+\frac{1}{2\pi}\sum_{j=3}^{n}\lg\frac{m_j(q^*)}{m_j(q)}\int_{C_q^{(j)}} \frac{\partial F(\omega, w_q; q)}{\partial \nu}\, ds + ic\ (w_q = f(z, q)).$$

The *harmonic measure* of $C_q^{(j)}$ at w_q with respect to $D_q + \Gamma_q$ is given by

$$P_j(w_q; q) = \frac{1}{2\pi}\int_{C_q^{(j)}} dH(\omega, w_q; q) = \frac{1}{2\pi}\int_{C_q^{(j)}} \frac{\partial G(\omega, w_q; q)}{\partial \nu}\, ds.$$

If we denote by $Q_j(w_q; q)$ a harmonic function conjugate to $P_j(w_q; q)$, then

$$R_j(w_q; q) \equiv P_j(w_q; q) + iQ_j(w_q; q) = \frac{1}{2\pi}\int_{C_j} \frac{\partial F(\omega, w_q; q)}{\partial \nu}\, ds,$$

and hence the equation obtained above may be written also in the form

$$\lg \frac{f(z,q^*)}{f(z,q)} = \frac{1}{2\pi}\int_{\beta_0(q,q^*)}^{\beta_1(q,q^*)} \lg|\Phi(e^{i\varphi})|\frac{\partial F(e^{i\varphi}, w_q; q)}{\partial \nu}\, d\varphi + R_2(w_q; q)\lg\frac{q^*}{q}$$

$$+\sum_{j=3}^{n} R_j(w_q; q)\lg\frac{m_j(q^*)}{m_j(q)} + ic.$$

In order to eliminate the constant c, we snbstitute $z = Q$ and hence $w_q = q$ in the last equation. Subtracting the thus obtained equation from the last equation itself, we obtain finally

$$\lg\frac{f(z,q^*)}{f(z,q)}-\lg\frac{q^*}{q}=\frac{1}{2\pi}\int_{\beta_0(q,q^*)}^{\beta_1(q,q^*)}\lg|\Phi(e^{i\varphi})|\Big(\frac{\partial F(e^{i\varphi},w_q;q)}{\partial\nu}-\frac{\partial F(e^{i\varphi},q;q)}{\partial\nu}\Big)d\varphi$$

$$+(R_2(w_q;q)-1)\lg\frac{q^*}{q}+\sum_{j=3}^{n}R_j(w_q;q)\lg\frac{m_j(q^*)}{m_j(q)},$$

since we may put $R_2(q;q)=1$ and $R_j(q;q)=0$ for $3\leqq j\leqq n$.

On the other hand, applying the Cauchy's integral theorem to a branch of $(1/w_q)\lg(\Phi(w_q)/w_q)$ regular and one-valued in $D_q+\Gamma_q$, we get

$$0=\Re\frac{1}{2\pi i}\int\lg\frac{\Phi(\omega)}{\omega}\frac{d\omega}{\omega}=\frac{1}{2\pi}\int\lg\left|\frac{\Phi(\omega)}{\omega}\right|d\arg\omega$$

$$=\frac{1}{2\pi}\int_{\beta_0(q,q^*)}^{\beta_1(q,q^*)}\lg|\Phi(e^{i\varphi})|d\varphi-\frac{1}{2\pi}\int_0^{2\pi}\lg\frac{q^*}{q}d\varphi+\frac{1}{2\pi}\sum_{j=3}^{n}\int_{C_q^{(j)}}\lg\frac{m_j(q^*)}{m_j(q)}d\arg\omega$$

$$=\frac{1}{2\pi}\int_{\beta_0(q,q^*)}^{\beta_1(q,q^*)}\lg|\Phi(e^{i\varphi})|d\varphi-\lg\frac{q^*}{q},$$

or

$$\frac{1}{2\pi}\int_{\beta_0(q,q^*)}^{\beta_1(q,q^*)}\lg|\Phi(e^{i\varphi})|d\varphi=\lg\frac{q^*}{q}.$$

Since $\beta_0(q,q^*)$ and $\beta_1(q,q^*)$ both tend to $\arg\bar{\gamma}(q)=-\theta(q)$ as $q^*\to q$, we conclude from the relations above obtained, by performing the limit-process $q^*-q\to 0$,

$$\frac{\partial\lg f(z,q)}{\partial\lg q}=\frac{\partial F(\bar{\gamma}(q),w_q;q)}{\partial\nu}-\frac{\partial F(\bar{\gamma}(q),q;q)}{\partial\nu}$$

$$+R_2(w_q;q)+\sum_{j=3}^{n}R_j(w_q;q)\frac{d\lg m_j(q)}{d\lg q},$$

the fundamental differential equation which has been desired. The integration of this equation with initial condition $f(z,Q)=z$ *will yield the mapping function* $f(z)=f(z,Q_0)$.

3. In the last step of our preceding argument it will further be required to ensure that $f(z,q)$ and $m_j(q)$ $(3\leqq j\leqq n)$ are all *differentiable* with respect to q; the fact which will be proved in the following lines.

In case with no circular slit ($n=2$) the differentiability of $f(z,q)$ with respect to q is already known.[6] In order to prove the same fact also in general case by induction, we suppose that n be greater than two and in $(n-1)$-ply connected case the function corresponding to $f(z,q)$ be known to be differentiable with respect to q. We

6) Cf. loc. cit.[3]

now construct an $(n-1)$-ply connected domain $\hat{D}_{Q_0}$, contained in D_{Q_0}, by cutting D_{Q_0} along a Jordan cross-cut which connects an end-point of a circular slit, e.g. $C_{Q_0}^{(n)}$ say, with the interior end-point of the slit Γ_{Q_0}. Denote by $\hat{D}_Q$ the domain which is contained in D_Q and corresponds to $\hat{D}_{Q_0}$ by the mapping $w=f(z, Q_0)$, $w=w_{Q_0}$, $z=w_Q$, namely we put $\hat{D}_{Q_0}=f(\hat{D}_Q, Q_0)$. The circular slit annulus $\hat{D}_{\hat{Q}}$ on the $w_{\hat{Q}}$-plane, conformally equivalent to $\hat{D}_Q$, can then be connected with $\hat{D}_{Q_0}$ by a family of the structure

$$\hat{f}(w_{\hat{Q}}, q)=\begin{cases}\hat{f}(w_{\hat{Q}}, q) & (\hat{Q}\geqq q\geqq Q),\\ f(\hat{f}(w_{\hat{Q}}, Q), q) & (Q\geqq q\geqq Q_0).\end{cases}$$

By our assumption of the induction, any function of this family and, in particular, the function

$$f(\hat{f}(w_{\hat{Q}}, Q), q) \quad (Q\geqq q\geqq Q_0,\ w_{\hat{Q}}\in \hat{D}_{\hat{Q}}, \hat{f}(w_{\hat{Q}}, Q)\in \hat{D}_Q)$$

is differentiable with respect to q. Hence, the same is valid for $f(z, q)$ $(z\in \hat{D}_Q)$. Now, $D_Q-\hat{D}_Q$ being the image of the Jordan cross-cut $D_{Q_0}-\hat{D}_{Q_0}$ laid in D_{Q_0}, the freedom of its choice——$f(z, q)$ $(Q\geqq q\geqq Q_0)$ remains invariant for any such a choice——shows that $f(z, q)$ $(z\in D_Q)$ is also differentiable with respect to the parameter q.

Remembering the equation just prevenient to the limit-process we have been performed, we see that in case $n=3$ the function $m_3(q)$ is differentiable. In case $n>3$ may also, by a similar argument as above, be reduced to the triply-connected case where only one circular slit $C_q^{(j)}$ exists, and hence the differentiability of $m_j(q)$ is thus surely established.

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