46. Stochastic Processes Built From Flows.*

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By virtue of the theory of semi-groups due to E. Hille[1] and the present author[2], we may construct stochastic processes in a separable measure space R from flows in R.

1. A flow in R is a one-parameter group $\{F_t\}$ of equi-measure transformations in R which is continuous in the sense that $f(F_t\cdot x)$, $f(x)\in L_p(R)$ $(1\leqq p<\infty)$, is strongly continuous in t. Thus the flow $\{F_t\}$ induces a one-parameter group $\{T_t\}$ of linear operators in $L_p(R)$:

(1.1) $$(T_t f)\ (x) = f(F_t\cdot x),\ f\in L_p(R),$$

(1.2) $$T_t T_s = T_{t+s},\ T_0 = I \quad \text{(the identity)},$$

(1.3) $$\text{strong}\lim_{t\to t_0} T_t f = T_{t_0} f.$$

Each T_t is a transition operator in $L_p(R)$:

(1.4) $f(x)\geqq 0$ implies $(T_t f)\ (x)\geqq 0$ and $\int_R f(x)\,dx=\int_R (T_t f)\ (x)\,dx$.

By the semi-group theory, $\{T_t\}$ admits infinitesimal generator A:

(1.5) $$\begin{cases} Af = \text{strong}\lim\limits_{t\downarrow 0} \dfrac{T_t - I}{t} f \text{ for those } f \text{ which are dense in } L_p(R), \\ T_t f = \exp(tA)f = \text{strong}\lim\limits_{n\to\infty} \exp\left(nt\left[(I-n^{-1}A)^{-1}-I\right]\right) f, -\infty<t<\infty. \end{cases}$$

Since $(I-n^{-1}A)^{-1}$ exists as a transition operator for $n\gtrless 0$[3],

(1.6) $$(I-n^{-1}A^2)^{-1} = (I-\sqrt{n^{-1}}A)^{-1}\,(I+\sqrt{n^{-1}}A)^{-1}$$

exists as a transition operator. Hence A^2 is the infinitesimal generator of a one-parameter semi-group $\{S_t\}$ of transition operators:

(1.7) $$S_t f = \exp(tA^2)f,\ 0\leqq t<\infty.$$

Thus the Fokker-Planck's equation in a Riemannian space R:

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* The following result was, under somewhat more restricted conditions and without proof, reported in May 1950 to the Conference in Probability of the International Congress of Mathematicians. Similar result with an interesting formulation was also obtained by Dr. Kiyosi Itô, by virtue of his theory of stochastic differential equations. See his paper in the same issue of this Proceedings.

1) Functional Analysis and Semi-groups, New York (1948).

2) On the differentiability and the representation of one-parameter semi-group of linear operators, J. Math. Soc. Japan, 1 (1948).

3) Since $\{T_t\}$ is a group (not only a semi-group).

(1.8) $\partial f(t, x)/\partial t = A^2 f(t, x), f(o, x) = f(x) \in L_1(R), t \geqq 0$

is integrable stochastically if

(1.9) $$A = p^i(x)\frac{\partial}{\partial x^i}$$

is the infinitesimal transformation of a one-parameter Lie group of equi-measure transformations in R.

2. Let a Riemannian space R admit several flows:

(2.1) $$\exp(tA_k), \ -\infty < t < \infty, \ (k = 1, 2, \ldots, m),$$
$$A_k = p^{ki}(x)\frac{\partial}{\partial x^i}.$$

Then, if the matrix (h^{ij}) is symmetric and positive definite, the operator

(2.2) $$C = h^{ij} A_i A_j$$

is the infinitesimal generator of a one-parameter semi-group of transition operators:

(2.3) $$\exp(tC), \ t \geqq 0.$$

Proof. C is, as an operator in $L_2(R)$, symmetric and negative definite:

(2.4) $\begin{cases} (Cf, g) = (f, Cg) \text{ and } (Cf, f) \leqq 0 \text{ for twice (continuously)} \\ \text{differentiable functions } f, g \text{ which are} \equiv 0 \text{ outside a compact} \\ \text{set of } R.\text{[4]} \end{cases}$

Hence[5] C admits self-adjoint extension $\tilde{C}$ which is also negative definite. Thus, for $n > 0$, $(I-n^{-1}\tilde{C})^{-1}$ exists as linear operator in $L_2(R)$ of norm $\leqq 1$. Therefore, for any $g \in L_1(R) \cap L_2(R)$, there exists uniquely determined $f \in L_2(R)$ such that

(2.5) $$(I-n^{-1}\tilde{C}) f = g.$$

Let a compact set R_1 be so chosen that

(2.6) $$\int_{R-R_1} | g(x) | dx < \varepsilon.$$

Then we may find $f_\varepsilon(x)$ with the properties:

(2.7) $$f_\varepsilon(x) \equiv 0 \quad \text{for } x \in R-R_1,$$
$$\int_R | g(x)-g_\varepsilon(x) | \, dx < 2\varepsilon \text{ where } g_\varepsilon = (I-n^{-1}C) f_\varepsilon.$$

This we see from (2.6) and the vanishing of $g_\varepsilon(x)$ in $R-R_1$. Therefore

4) Since the divergences of the vectors $(p^{k1}(x), p^{k2}(x), \ldots, p^{kn}(x))$ vanish.

5) Cf. H. Freudenthal: Über die Friedrichssche Fortsetzung halbbeschränkter Hermitescher Operatoren, Proc. Amsterdam Acad., 39 (1936).

(2.8) $\begin{cases}\text{the range } \{(I-n^{-1}C)f;\ f\in L_1(R) \wedge (\text{the domain of } C)\} \text{ is dense} \\ \text{in } L_1(R).\end{cases}$

On the other hand C is of the form

$$C = b^{ij}(x)\frac{\partial^2}{\partial x^i\,\partial x^j} + a^i(x)\frac{\partial}{\partial x^i}\,, \tag{2.9}$$

where $b^{ij}(x)\xi_i\xi_j \geqq 0$. Hence we have, from

(2.10) $(I-n^{-1}C)f_\varepsilon = g_\varepsilon$ ($f_\varepsilon = g_\varepsilon = 0$ outside the compact set R_1),

(2.11) $\min f_\varepsilon(x) = f_\varepsilon(x_0) \geqq g_\varepsilon(x_0)$

and

$$\int_R f_\varepsilon(x)\,dx = \int_R f_\varepsilon(x)\,dx - n^{-1}\int_R (Cf_\varepsilon)(x)\,dx = \int_R g_\varepsilon(x)\,dx\text{[6]}. \tag{2.12}$$

From (2.10), (2.11) add (2.12) we see that, if $n > 0$, $(I-n^{-1}\bar{C})^{-1}$ exists as a transition operator in $L_1(R)$[7]. Here $\bar{C}$ denotes a closed extension of C. Thus $\exp(t\bar{C})$ is, for $t \geqq 0$, a one-parameter semi-group of transition operators.

3. Let, in particular, the group G of motions of R be a compact semi-simple Lie group with the infinitesimal transformations $X_1, X_2, \ldots, X_m$ transitive on R. Then the so-called Casimir operator

(3.1) $C = h^{ij}X_iX_j$, where $(h^{ij}) = (h_{ij})^{-1}$, $h_{ij} = c_{i\rho}^{\sigma}\, c_{j\sigma}^{\rho}$, $[X_i, X_j] = c_{ij}^{k}\, X_k$

is commutative with every X_i. By the compactness of G, (h^{ij}) is a (truly) positive definite symmetric matrix. Therefore, by the result in 2, C is the infinitesimal geperator of a one-parameter semi-group $\{\exp(tC)\}$, $t \geqq 0$, of transition operators. Moreover, by the commutativity

$$[C, X_i] = 0 \qquad (i = 1, 2, \ldots, m), \tag{3.2}$$

$\{\exp(tC)\}$ defines a temporally and spatially homogeneous "continuous" stochastic process in R[8] —— a Brownian motion in the homogeneous space R.

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6) By the same reason as stated in 4).

7) Cf. K. Yosida: Integration of Fokker-Planck's equation in a compact Riemannian space, Arkiv för Matematik, **1**, No. 2 (1949).

8) Cf. K. Yosida: Brownian motion on the surface of the 3-sphere, Ann. of Math. Statistics, **20**, No. 2 (1949).