70. *A Proof of the Hahn-Birkhoff Theorem. Notes on Banach Space (X).*

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In the preceding paper [2], it is pointed out that the Hahn-Birkhoff decomposition theorem on a strictly monotone Banach lattice can be extended to a complete Banach lattice having an order-continuous linear functional and can be proved basing on the idea of G. Birkhoff [1; 119] with a few modification. In this note we will give an alternative simpler proof of the theorem using Zorn's Lemma. In the sequel, to save the space, we use the notations and terminologies of G. Birkhoff's text book [1] without any explanations.

Let E be a complete Banach lattice having an *order-continuous* linear functional f, that is, $f(x_\alpha)$ converges to zero for any $x_\alpha \downarrow 0$, where we mean $x_\alpha \downarrow 0$ if $\{x_\alpha\}$ is a Moore-Smith set of E with $x_\alpha \leqq x_\beta$ for $\alpha > \beta$ and the infimum of x_α's is zero. After G. Birkhoff, the *positive, negative* and *null ideals* are defined as follows:

$$P = \{ x \mid 0 < y \leqq |x| \rightarrow f(y) > 0 \} ,$$
$$M = \{ x \mid 0 < y \leqq |x| \rightarrow f(y) < 0 \} ,$$
$$N = \{ x \mid 0 < y \leqq |x| \rightarrow f(y) = 0 \} .$$

These ideals are linearly independent normal subspaces as proved by G. Birkhoff. But by the sake of the completeness we will give it in full. Suppose that x and y belong to P. Then $0 < z \leqq |x+y|$ implies $0 < z \leqq |x| + |y|$, whence $z = x' + y'$ where $0 \leqq x' \leqq |x|$, $0 \leqq y' \leqq |y|$ and either x' or y' is strictly positive, and so $f(z) = f(x') + f(y')$ is strictly positive. That is, P is a subspace. Moreover, if x belongs to P and $|y| \leqq |x|$ then $0 < z \leqq |y|$ implies $0 < z \leqq |x|$, and so $f(z)$ is strictly positive, whence y belongs to P, that is, P is a normal subspace. Similarly, we may prove that M and N are also normal subspaces. To prove the linear independence, it is sufficient to show $P \frown M = M \frown N = N \frown P = 0$, or equivalently, each positive element belongs to at most one of such ideals. But this is plain, since x belongs to P if and only if $f(x)$ is strictly positive. This proves the above statement. (It is to be noted, that in the above the order-continuity of the functional is not used essentially, whence it is true for any linear functional).

Now we will use the order-continuity. Then *the above three ideals become complemented normal subspaces* in the sense of G. Birkhoff [1; 111]. By a remark of G. Birkhoff [1; 121], to prove this, it is sufficient to show that each order-bounded set of P, M, and N has its supremum in P, M, and N respectively. Now, suppose that S be the set of all elements of P with $0 \leqq x \leqq a$ where a is an arbitrary positive element of E. Then S has the Moore-Smith property and order-converges to its supremum b. Hence $x \frown y$ with x in S order-converges to y for any y with $0 < y \leqq b$ by the order-continuity of the lattice operations. Since $x \frown y$ is strictly positive for any x with $x \geqq x'$ and $x \frown y$ belongs to P, we have $f(x \frown y) \geqq f(x' \frown y) > 0$, that is, b belongs to P by the definition. Similar arguments show that M and N are also complemented normal subspaces.

Under these considerations the theorem reads as follows:

THEOREM (Hahn-Birkhoff). *By means of an order-continuous linear functional, a complete Banach lattice is decomposed into the direct union of positive, negative and null ideals.*

Proof: Suppose the contrary. Then there exists a positive element a with $f(a) > 0$, which belongs to the intersection of the complement normal subspaces of the above three ideals. Let S be the set of all x with $0 < x < a$ and $f(x) \geqq f(a)$. If S is void, then $0 < x < a$ implies $0 < f(x) < f(a)$ since $0 < a - x < a$ holds, and so a belongs to the positive ideal and a contradiction. Hence S is non-void. If S contains a decreasing simply ordered subset L with the infimum b, then the elements of L order-converges to b, whence by the order-continuity of the functional f it holds $f(b) \geqq f(a)$, and so b belongs to S by the assumption, whence S is inductively ordered, and consequently by Zorn's Lemma S has at least one minimal element c. On the other hand, S has the Moore-Smith property, whence S contains at most one minimal element, that is, c is the infimum of S. Hence $0 < x < c$ implies $f(x) > 0$ since otherwise $c - x$ belongs to S and it contradicts to the minimality of c. This is nothing but c belongs to the positive ideal, which is a contradiction to our hypothesis. This completes the proof of the theorem.

To conclude the note it may be pointed out that the existence of the norm and the metric completeness of the space are not essentially needed in the proof of the theorem. Hence we can state the theorem for a complete vector lattices having an order-continuous additive and homogeneous functional.

REFERENCES.

1. G. Birkhoff: *Lattice Theory*, New York, 1940.
2. M. Nakamura: *Notes on Banach Space* (IX), Tôhoku Mathematical Journal, 1 (1949), 100–108.