17. An Alternative Proof of Liber's Theorem.

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§ 1. Introduction.

In Mathematical Review 11 (1950), Prof. M. S. Knebelmann communicated the following results of Liber (Doklady Akad. Nauk SSSR (N.S), 66 (1949)) concerning the structure of affinely connected and Riemannian spaces with one-parametric holonomy groups.

Theorem A. *Suppose that the holonomy group H of a given affinely connected space A_n be a one parametric group. If we denote the symbol of the infinitesimal transformation of H by $Xf = a^i_j x^j \frac{\partial f}{\partial x^i}$ (a^i_j : const.), then the rank of the matrix $\| a^i_j \|$ is at most 2.*

Theorem B. *Suppose that the holonomy group H of a given Riemannian space V_n be a one parametric group. Then V_n admits $(n-2)$ parallel vector fields which are independent each other. Accordingly, V_n is the product space of a two dimensional Riemannian space and an $(n-2)$ dimensional Euclidean space.*

I shall give here alternative proofs of Liber's theorems. Although I can not see his paper, it is certain that my proof is quite different from his original proof. Perhaps my proof will be more geometrical than his proof.

§ 2. Riemannian spaces.

We shall first state *Cartan's Lemma.* Suppose that the holonomy group of a nonholonomic space E with the fundamental group G be g. Then we can choose frames at each point of E so that the connexion of the space in consideration is analytically the same as those of a space with the fundamental group g.

When we are going to apply this Lemma to Riemannian and affinely connected spaces, we must note that the word "holonomy group" is used in different senses in introduction and in Cartan's Lemma. The holonomy group in introduction is the so called "homogeneous holonomy group" that is the group of linear homogeneous transformations belonging to the holonomy group in ordinary sense.

Now, suppose that the holonomy group of a given Riemannian space V_n be a one parametric continuous group H (of rotations). Then on account of Cartan's Lemma, we can choose frames at each point of V_n so that the connexion of V_n is analytically the same as that of a space with the fundamental group H. We shall assume that orthogonal frames at each point of V_n are so chosen. If we denote by

$$(1) \qquad Xf = a_{ij}\, x^i \frac{\partial f}{\partial x^j}, \qquad a_{ij} = -a_{ji},$$

(where a_{ij} are constants) the generator of H and the equations of definition of the connexion of V_n by

$$dP = \omega^i e_i, \qquad de_i = \omega_{ij}\, e_j,$$

then we obtain the following relation:

$$(2) \qquad \omega_{ij} = a_{ij}\,\omega,$$

(ω is a Pfaffian which appears as the proportionality factor). Putting the last equation in

$$\Omega_{ij} = -\omega'_{ij} + [\omega_{ik}\,\omega_{jk}],$$

we get

$$(3) \qquad \Omega_{ij} = -a_{ij}\omega',$$

where dashes mean the exterior derivation of Pfaffians (differential forms of rank 1). Hence if we put

$$(4) \qquad \Omega_{ij} = A_{ijkl}\,[\omega^k \omega^l],$$

wet get

$$(5) \qquad A_{ijkl} = a_{ij}\, B_{kl},$$

where B_{kl} are components of a quantity defined by

$$\omega' = -B_{kl}\,[\omega^k \omega^l].$$

On the other hand, A_{ijkl}'s satisfy the following relation:

$$(6) \qquad A_{ijkl} = A_{klij},$$

$$(7) \qquad A_{ijkl} + A_{iklj} + A_{iljk} = 0.$$

If we use (5), (6) and (7), we can easily see that the following relations hold good:

$$(8) \qquad A_{ijkl} = a_{ij}\, a_{kl}\, B$$

$$a_{ij}a_{kl}+a_{ik}a_{lj}+a_{il}a_{jk}=0. \tag{9}$$

(9) shows that the bivector a_{ij} is simple. Hence if we perform suitable orthogonal transformations for orthogonal frames in consideration, we can give the generator Xf the following canonical expression:

$$Xf = x^1\frac{\partial f}{\partial x^2} - x^2\frac{\partial f}{\partial x^1}.$$

Accordingly, the holonomy group H fixes $(n-2)$ mutually independent directions. Consequently, we know by [2], that the following theorem holds good:

Theorem 1. *In order that a Riemannian space has one parametric holonomy group, it is necessary and sufficient that there exists a coordinate system such that the line element of the space in consideration reduces to the following canonical form:*

$$ds^2 = d\sigma^2 + (du^3)^2 + (du^4)^2 + \dots + (du^n)^2,$$

where we have put

$$d\sigma^2 = g_{ab}(u^c)du^a du^b \qquad (a, b, c = 1, 2).$$

Theorem 1 is essentially same as Theorem B in § 1.

§ 3. Affinely connected spaces.

Suppose that the holonomy group of a given affinely connected space A_n without torsion be a one parametric continuous group H (of central affine transformations). Then on account of Cartan's Lemma, we can choose frames at each point of A_n so that the connexion of A_n is analytically the same as that of a space with the fundamental group H. We shall assume that cartesian frames at each point of A_n are so chosen. If we denote by

$$Xf = a^i_j x^j \frac{\partial f}{\partial x^i} \tag{10}$$

(where a^i_j are constants) the generator of H and the equations of definition of the connexion of A_n by

$$dP = \omega^i e_i, \qquad de_j = \omega^i_j e_i,$$

then we obtain the following relation:

$$\omega^i_j = a^i_j \omega. \tag{11}$$

Putting the last equation in

$$\Omega^i_j = -(\omega^i_j)' + [\omega^k_j \omega^i_k]$$

wet get

(12) $$\Omega^i_j = -a^i_j \omega'.$$

Hence if we put

(13) $$\Omega^i_j = A^i_{jkl}[\omega^k \omega^l],$$

we see that A^i_{jkl} has the following form:

(14) $$A^i_{jkl} = a^i_j B_{kl}, \qquad (B_{kl} = -B_{lk}).$$

However as A^i_{jkl}'s satisfy the relation

$$A^i_{jkl} + A^i_{klj} + A^i_{ljk} = 0,$$

we get finally the following relation:

(15) $$a^i_j B_{kl} + a^i_k B_{lj} + a^i_l B_{jk} = 0.$$

Now, let us make some convention. In the first place we shall use two sets of indices i, j, k, l and $\alpha, \beta, \gamma, \delta$, both take values $1, 2, \ldots, n$. But for the latter, we assume that they take some fixed values in each proposition. In the second place we shall denote the equation (15) for $j=\alpha$, $k=\beta$, $l=\gamma$ by $E_{\alpha\beta\gamma}$. In the third place, when a set of $\frac{r(r-1)}{2}$ equations $B_{ab}=0$ $(a, b=1, 2\ldots, r)$ hold simultaneously, we shall say that "the assumption $A_{12\ldots r}$ is satisfied or holds good," and when at least one of B_{ab} does not vanish we shall say that "$A_{12\ldots r}$ does not satisfied" or "$\bar{A}_{12\ldots r}$ holds good." Finally, let us consider a^i_α as homogeneous coordinates of a point a_α in projective $(n-1)$ space P_{n-1}.

Then we can easily obtain from (15) the following

Lemma L_1. If $\bar{A}_{12\alpha}$ holds good the rank of the $(3, n)$ matrix $\|a^i_1 a^i_2 a^i_\alpha\|$ is at most 2. In other words, the three points a_1, a_2, a_α are collinear or some of them does not represent an actual point (for example $a^i_\alpha \equiv 0$). On the contrary, if $A_{12\alpha}$ is satisfied, we can conclude nothing about its rank.

The remaining part of this paragraph is devoted to the proof of Theorem A. If the rank of the (n, n) matrix $\|a^i_j\| \leq 1$, there remains nothing to prove. Hence we shall assume hereafter that the rank of $\|a^i_j\|$ is at least 2, that is, there exist at least two distinct actual points among $a_1, \ldots. a_n$ in P_{n-1}. We can assume without any loss of generality that a_1 and a_2 are distinct actual points.

First, if we assume that $\bar{A}_{123}$ holds, then we can easily see, in virtue of E_{123} and our hypothesis on the point a_1, a_2, that $B_{12} \neq 0$. Hence $a^i_3 \equiv 0$ or the point a_3 lies on the line joining a_1

and a_2. An analogous result follows also from the assumption $\bar{A}_{124}$. Accordingly, if $\bar{A}_{123}$ and $\bar{A}_{124}$ hold good, then the rank of the $(4, n)$ matrix $\| a_1^i a_2^i a_3^i a_4^i \|$ is equal to 2.

Now, it does not happen that A_{123} and $\bar{A}_{124}$ are satisfied. For, these assumptions lead us, on account of E_{124} immediately to a contradiction. Analogous fact holds good also for $\bar{A}_{123}$ and A_{124}.

Lastly, if A_{123} and A_{124} hold simultaneously, then $B_{34} = 0$. For if we assume $B_{34} \neq 0$, then, by virtue of E_{134} and E_{234}, we meet a contradiction. Accordingly, we get the following

Lemma L_2. If the assumption $\bar{A}_{1234}$ is satisfied, then the rank of the matrix $\| a_1^i a_2^i a_3^i a_4^i \|$ is equal to 2. On the contrary, if A_{1234} is satisfied we can conclude nothing about its rank.

Noting the analogy between Lemmas L_1 and L_2 we want to prove Theorem A by mathematical induction. Let us first assume that Lemmas $L_1, L_2, \ldots$ and L_{r-3} are true and prove L_{r-2}. We can first see that the following proposition is true:

(i) $\bar{A}_{12\ldots r-1}$ holds good and the rank of the matrix $\| a_1^i a_2^i \ldots a_{r-1}^i \|$ is equal to 2, or

(i)′ $A_{12\ldots r-1}$ holds good, and we can conclude nothing about the rank of the matrix.

In the same way, we see

(ii) $\bar{A}_{12\ldots r-2, r}$ holds good and the rank of the matrix $\| a_1^i a_2^i \ldots a_{r-2}^i a_r^i \|$ is equal to 2, or

(ii)′ $A_{12\ldots r-2, r}$ holds good and we can conclude nothing about the rank of the matrix.

Analogous to the proof of Lemma L_2, we can easily show that only combinations (i), (ii) and (i)′, (ii)′ are possible and they lead to the

Lemma L_{r-2}. If $\bar{A}_{12\ldots r}$ holds good then the rank of the matrix $\| a_1^i a_2^i \ldots a_r^i \|$ is equal to 2. On the contrary, if $A_{12\ldots r}$ holds good, we can conclude nothing about its rank.

On the other hand, the assumption $A_{12\ldots n}$ is equivalent that the affinely connected space in consideration is flat. Hence there remains only the case where the rank $\| a_j^i \|$ is equal to 2. Consequently the proof of Theorem A is finished.

§ 4.

Let us denote the characteristic roots of the equation

$$| a_j^i - \rho \delta_j^i | = 0$$

by $\rho_1, \rho_2, \ldots, \rho_n$. We assume that equal roots occupy consecutive position in this arrangement of roots. Then, performing a suitable linear coordinate transformation, we can reduce the matrix $\|a^i_j\|$ to the following canonical form:

$$\left\| \begin{matrix} \rho_1 & e_1 & 0 & \cdots & \cdots & 0 \\ 0 & \rho_2 & e_2 & \ddots & & \vdots \\ \vdots & \ddots & \rho_3 & e_3 & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ \vdots & & & \ddots & \rho_{n-1} & e_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & \rho_n \end{matrix} \right\| ,$$

where e_i is 0 or $+1$ and when $e_i = 1$, $\rho_i = \rho_{i+1}$ (See [3]). However as the rank of the matrix is at most 2, only the following canonical forms are possible:

(I) The case where all e_i vanish. In this case the canonical form reduces to one of the type (I_0) or (I) of the next table of matrices.

(II) The case where just one of e_i does not vanish. In this case the canonical form reduces to one of the type (II_0), (II_1) or (II_2).

(III) The case where just two of e_i does not vanish. In this case the canonical from reduces to the type (III).

Table of Matrices.

$$(I_0)\ \left\| \begin{array}{c|c} \rho & 0 \\ \hline 0 & 0 \end{array} \right\| \qquad (I)\ \left\| \begin{array}{cc|c} \rho_1 & 0 & 0 \\ 0 & \rho_2 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array} \right\|$$

$$(II_0)\ \left\| \begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array} \right\| \qquad (II_1)\ \left\| \begin{array}{cc|c} \rho & 1 & 0 \\ 0 & \rho & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array} \right\|$$

$$(II_2)\ \left\| \begin{array}{ccc|c} 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \rho & \\ \hline \multicolumn{3}{c|}{0} & 0 \end{array} \right\| \qquad (III)\ \left\| \begin{array}{ccc|c} 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \\ \hline \multicolumn{3}{c|}{0} & 0 \end{array} \right\|$$

Consequently, the generators of the corresponding holonomy groups are given as follows:

$$(\mathrm{I}_0) \qquad x^1\frac{\partial f}{\partial x^1},$$

$$(\mathrm{I}) \qquad x^1\frac{\partial f}{\partial x^1}+a\,x^2\frac{\partial f}{\partial x^2}, \qquad (a: \text{const.} \neq 0)$$

$$(\mathrm{II}_0) \qquad x^2\frac{\partial f}{\partial x^1},$$

$$(\mathrm{II}_1) \qquad a\left(x^1\frac{\partial f}{\partial x^1}+x^2\frac{\partial f}{\partial x^2}\right)x^2+x^2\frac{\partial f}{\partial x^1},$$

$$(\mathrm{II}_2) \qquad x^2\frac{\partial f}{\partial x^1}+a\,x^3\frac{\partial f}{\partial x^2},$$

$$(\mathrm{III}) \qquad x^2\frac{\partial f}{\partial x^1}+x^3\frac{\partial f}{\partial x^2}.$$

Theorem 2. *If the holonomy group of any affinely connected space A_n without torsion is one parametric group, then A_n admits at least $(n-2)$ mutually independent parallel vector fields.*

(The converse is not true in general).

Proof. We can easily see from the canonical forms of the matrix $\|a^i_j\|$ that the connexion of the space in consideration is given by one of the following equations:

$$dP = \omega^i e_i$$

$$(\mathrm{I}_0) \qquad de_1 = \omega^1_1 e_1,\ de_2 = 0,\ \ldots\ldots,\ de_n = 0,$$

$$(\mathrm{I}) \qquad de_1 = \omega^1_1 e_1,\ de_2 = \omega^2_2 e_2,\ de_3 = 0,\ \ldots,\ de_n = 0,$$

$$(\mathrm{II}_0) \qquad de_1 = 0,\ de_2 = \omega^1_2 e_1,\ de_3 = 0,\ \ldots\ldots,\ de_n = 0,$$

$$(\mathrm{II}_1) \qquad de_1 = \omega^1_1 e_1,\ de_2 = \omega^1_2 e_1 + \omega^2_2 e_2\, de_3 = 0,\ \ldots,\ de_n = 0,$$

$$(\mathrm{II}_2) \qquad de_1 = 0,\ de_2 = \omega^1_2 e_1,\ de_3 = \omega^2_3 e_3,\ de_4 = 0,\ \ldots,\ de_n = 0,$$

$$(\mathrm{III}) \qquad de_1 = 0,\ de_2 = \omega^1_2 e_1,\ de_3 = \omega^2_3 e_2,\ de_4 = 0,\ \ldots,\ de_n = 0.$$

Hence, we can easily conclude that the connexions of the type (I), (II) and (III) admit $(n-1)$ and $(n-2)$ mutually independent parallel vector fields respectively. Q. E. D.

From the last theorem we can immediately obtain Theorem B as its corollary. For the one parametric group H of the type (III) there exists non-singular quadratic forms which are invariant under H. Hence the A_n may be regarded as a Riemannian space. (Cf. M. Abe [4]).

We can also derive the canonical forms of parameters of affine connexions in consideration, but we omit them, for they are somewhat complicated for types (II_1), (II_2) and (III).

References.

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