16. On the Simple Extension of a Space with Respect to a Uniformity. I.

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In the present and the next notes we shall develop a general theory concerning the simple extension of a space with respect to a uniformity. As special cases we obtain various topological extensions of spaces such as completions of uniform spaces in the sense of A. Weil[1] (or more generally in the sense of L. W. Cohen[2]) and the bicompact extensions of T-spaces due to N. A. Shanin[3] (a generalization of Wallman's bicompactification).

§ 1. **Definitions.** In the present note we say that R is a *space*, if R is an aggregate of "points" and to each subset A of R there corresponds a set $\bar{A}$, called the closure of A, with the following properties:

1) $A \subset \bar{A}$, 2) $\bar{\bar{A}} = \bar{A}$,

3) $A \subset B$ implies $\bar{A} \subset \bar{B}$, 4) $\bar{0} = 0$.

Thus R is a neighbourhood space such that we can take as a basis of neighbourhoods of a point p a family of open sets containing p. As is well known a space which satisfies the additivity of the closure operation: $\overline{A+B} = \bar{A}+\bar{B}$ is a T-space.

Let R be a space. A collection $\{\mathfrak{U}_\alpha ; \alpha \in \Omega\}$ of open coverings of R is called a *uniformity*. Two uniformities $\{\mathfrak{U}_\alpha\}$ and $\{\mathfrak{V}_\lambda\}$ are called *equivalent*, if for any $\mathfrak{U}_\alpha \in \{\mathfrak{U}_\alpha\}$ there exists a covering $\mathfrak{V}_\lambda \in \{\mathfrak{V}_\lambda\}$ which is a refinement of $\mathfrak{U}_\alpha$, and conversely for any $\mathfrak{V}_\lambda$ there exists $\mathfrak{U}_\beta \in \{\mathfrak{U}_\alpha\}$ such that $\mathfrak{U}_\beta$ is a refinement of $\mathfrak{V}_\lambda$. We say that a uniformity $\{\mathfrak{U}_\alpha ; \alpha \in \Omega\}$ *agrees with the topology*, if it satisfies the condition:

(A) $\{S(p, \mathfrak{U}_\alpha) ; \alpha \in \Omega\}$ is a basis of neighbourhoods at each point p of R.

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1) A. Weil: Sur les espaces a structure uniforme et sur la topologie générale, Actualites Sci. Ind. **551**, 1937; J. W. Tukey: Convergence and uniformity in topology, 1940.

2) L. W. Cohen: On imbedding a space in a complete space, Duke Math. J. **5** (1939), 174–183.

3) N. A. Shanin: On special extensions of topological spaces, Doklady URSS, **38** (1943), 3–6; On separation in topological spaces, ibid., 110–113; On the theory of bicompact extensions of topological spaces, ibid., 154–156. These papers are not yet accessible to us. We knew the results by Mathematical Reviews.

Here we denote by $S(A, \mathfrak{U})$ the sum of all the sets of a covering $\mathfrak{U}$ intersecting a subset A of R[4]. A uniformity $\{\mathfrak{U}_\alpha ; \alpha \in \Omega\}$ is called a *T-uniformity*, if it satisfies the condition:

(B) For any $\alpha, \beta \in \Omega$ there exists $\gamma \in \Omega$ such that $\mathfrak{U}_\gamma$ is a refinement of $\mathfrak{U}_\alpha$ and $\mathfrak{U}_\beta$.

According as $\{\mathfrak{U}_\alpha ; \alpha \in \Omega\}$ satisfies the condition:

(C) For any $\alpha \in \Omega$ there exists $\lambda(\alpha) \in \Omega$ such that for each set $U \in \mathfrak{U}_{\lambda(\alpha)}$ we can determine a set U_α of $\mathfrak{U}_\alpha$ and $\delta = \delta(\alpha, U) \in \Omega$ so that $S(U, \mathfrak{U}_\delta) \subset U_\alpha$,

or the condition:

(D) For any $\alpha \in \Omega$ there exists $\lambda(\alpha) \in \Omega$ such that, for every set U of $\mathfrak{U}_{\lambda(\alpha)}$, $S(U, \mathfrak{U}_{\lambda(\alpha)})$ is contained in some set U_α of $\mathfrak{U}_\alpha$,

the uniformity $\{\mathfrak{U}_\alpha\}$ is called *regular* or *completely regular*. The condition (D) states that $\mathfrak{U}_\alpha$ has a star-refinement $\mathfrak{U}_{\lambda(\alpha)}$[4]. A completely regular uniformity is always regular. A space possessing a uniformity which agrees with the topology is a *uniform space*.

Remark. A uniform space in the sense of A. Weil and J. W. Tukey[5] is a T_1-space which has a completely regular T-uniformity agreeing with the topology. L. W. Cohen considered a T_1-space R such that for any point p and any element α of a set Ω of indices there is defined an open neighbourhood $V_\alpha(p)$ of p with the following properties: 1) $\{V_\alpha(p) ; \alpha \in \Omega\}$ is a basis of neighbourhoods at p, and 2) for $p \in R$ and for α there exist $\lambda(\alpha) \in \Omega$ and $\delta(p, \alpha) \in \Omega$ such that $V_{\delta(p,\alpha)}(q) \cdot V_{\lambda(\alpha)}(p) \neq 0$ implies $V_{\delta(p,\alpha)}(q) \subset V_\alpha(p)$ for every point q of R.[6] If we put $\mathfrak{B}_\alpha = \{V_\alpha(p) ; p \in R\}$ and construct all the finite intersections of the coverings $\mathfrak{B}_\alpha (\alpha \in \Omega)$, it is easily seen that the set of these coverings defines a regular T-uniformity agreeing with the topology.

§ 2. **Uniformisable spaces.** A space R is called *weakly regular*[7], if for every open set U containing any point p of R we have $\bar{p} \subset U$. R is called *regular*, if for any neighbourhood U of p there exists an open set H such that $p \in H$, $\bar{H} \subset U$. In case for any neighbourhood U of p there exists a real-valued bounded continuous function $f(x)$ such that $f(p) = 0$ and $f(x) = 1$ for $x \in R - U$, R is called *completely regular*.

Theorem 1. *In order that a space R possess a uniformity or a regular uniformity or a completely regular uniformity or a T-uni-*

4) J. W. Tukey: loc. cit.

5) Cf. A. Weil and J. W. Tukey: loc. cit., 1).

6) L. W. Cohen: loc. cit., 2).

7) N. A. Shanin: loc. cit., 3). Further a space satisfying the condition (D) of T. Inagaki is nothing but a weakly regular space as is shown by our Theorem 1 and his theorem in his paper: Sur les espaces à structure uniforme, Jour. Hokkaido Univ. Ser. 1, Vol. X (1943), p. 230.

formity, agreeing with the topology, it is necessary and sufficient that R be a weakly regular space, a regular space, a completely regular space or a weakly regular T-space respectively.

Proof. For the case of complete regularity we can prove the theorem similarly as in the case of A. Weil and J. W. Tukey[8]. Let R be a weakly regular space. Then the set $\{\mathfrak{U}_\alpha; \alpha \in \Omega\}$ of all the open coverings of R is a uniformity agreeing with the topology, since for an open set G containing a point p we have $S(p, \mathfrak{U}_\alpha) \subset G$, where $\mathfrak{U}_\alpha = \{G, R-\bar{p}\}$. Moreover, if R is regular, this uniformity is regular. Because for a covering $\mathfrak{U}_\alpha$ we can determine an open covering $\mathfrak{U}_{\lambda(\alpha)}$ such that the closure of each set of $\mathfrak{U}_{\lambda(\alpha)}$ is contained in some set of $\mathfrak{U}_\alpha$, aud hence for any set U of $\mathfrak{U}_{\lambda(\alpha)}$ we have $\bar{U} \subset$ some U_α, $U_\alpha \in \mathfrak{U}_\alpha$, and consequently, if we put $\mathfrak{U}_\delta = \{U_\alpha, R-\bar{U}\}$, we have $S(U, \mathfrak{U}_\delta) \subset U_\alpha$. If R is a T-space, then the above uniformity is clearly a T-uniformity.

The necessity of the condition follows readily from Lemma 1 below, whose proof is easy.

Lemma 1. *Let $\{\mathfrak{U}_\alpha; \alpha \in \Omega\}$ be a uniformity of a space R which agrees with the topology. Then for any subset A of R we have* $\bar{A} = \prod_{\alpha \in \Omega} S(A, \mathfrak{U}_\alpha)$.

Remark. A T_0-space is not always weakly regular. A weakly regular T_0-space is necessarily a T_1-space, as is shown by Theorem 1 and Lemma 1.

§ 3. The simple extension R^* of a space R with respect to a uniformity. Let $\{\mathfrak{U}_\alpha; \alpha \in \Omega\}$ be a uniformity of a space R. A family $\{X_\lambda; \lambda \in \Lambda\}$ of subsets of R is called *Cauchy family* (with respect to the uniformity $\{\mathfrak{U}_\alpha\}$), if it has the finite intersection property and satisfies the condition:

(1) For any $\alpha \in \Omega$ there exist a set $X_\lambda \in \{X_\lambda\}$ and $\beta \in \Omega$ and a set U_α of $\mathfrak{U}_\alpha$ such that
$$S(X_\lambda, \mathfrak{U}_\beta) \subset U_\alpha.$$

A Cauchy family $\{X_\lambda\}$ is said to be *vanishing*, if $\prod_{\lambda \in \Lambda} \bar{X}_\lambda = 0$. A Cauchy family $\{X_\lambda\}$ is said to be *equivalent* to another Cauchy family $\{Y_\mu\}$: written $\{X_\lambda\} \sim \{Y_\mu\}$, if for any $X_\lambda \in \{X_\lambda\}$ and any $\alpha \in \Omega$ there exist a set $Y_\mu \in \{Y_\mu\}$ and $\beta \in \Omega$ such that

(2) $$S(Y_\mu, \mathfrak{U}_\beta) \subset S(X_\lambda, \mathfrak{U}_\alpha).$$

Lemma 2. *If $\{X_\lambda\} \sim \{Y_\mu\}$, then $\{Y_\mu\} \sim \{X_\lambda\}$.*

Proof. For any $\alpha \in \Omega$ there exist $X_\lambda \in \{X_\lambda\}$, $\beta \in \Omega$ and $U_\alpha \in \mathfrak{U}_\alpha$ such that $S(X_\lambda, \mathfrak{U}_\beta) \subset U_\alpha$. By the assumption of Lemma 2 there

8) Cf. loc. cit., 1), in particular Tukey's book p. 58. It is to be noted that we do not assume the additivity of the closure operation which is not implied by the complete regularity.

exist $Y_{\mu_0} \in \{Y_\mu\}$ and $\gamma \in \Omega$ such that $S(Y_{\mu_0}, \mathfrak{U}_\gamma) \subset S(X_\lambda, \mathfrak{U}_\beta)$. Then we have $Y_\mu \cdot U_\alpha \neq 0$ for any $Y_\mu \in \{Y_\mu\}$, since $Y_\mu \cdot Y_{\mu_0} \neq 0$, and hence $S(X_\lambda, \mathfrak{U}_\beta) \subset U_\alpha \subset S(Y_\mu, \mathfrak{U}_\alpha)$. Thus we have $\{Y_\mu\} \sim \{X_\lambda\}$.

Lemma 3. *If* $\{X_\lambda\} \sim \{Y_\mu\}$ *and* $\{Y_\mu\} \sim \{Z_\nu\}$, *then* $\{X_\lambda\} \sim \{Z_\nu\}$.

Lemma 3 follows directly from the definition. Hence the equivalence of Cauchy families is an equivalence relation. It may happen that a non-vanishing Cauchy family is equivalent to a vanishing Cauchy family. In this connection we state the following lemma, which is an easy consequence of Lemma 1.

Lemma 4. *If* $\{\mathfrak{U}_\alpha\}$ *agrees with the topology and* $\{X_\lambda\} \sim \{Y_\mu\}$, *then* $\prod_\lambda \bar{X}_\lambda = \prod_\mu \bar{Y}_\mu$.

We consider the equivalence classes of vanishing Cauchy families; we denote the set of these classes by C. For any open set G of R we define the set G^* as a subset of $R+C$ as follows: a point $x \in C$ belongs to G^* if for any Cauchy family $\{X_\lambda\}$ of the class x there exist $X_\lambda \in \{X_\lambda\}$ and $\alpha \in \Omega$ such that $S(X_\lambda, \mathfrak{U}_\alpha) \subset G$[9], and a point x of R belongs to G^* if $x \in G$; that is,

(3) $G^* = G + \{x;\ \{X_\lambda\} \in x$ implies that $S(X_\lambda, \mathfrak{U}_\alpha) \subset G$ for some $X_\lambda \in \{X_\lambda\}$ and $\mathfrak{U}_\alpha\}$.

Then we have

Lemma 5. $G^* \cdot R = G,\quad 0^* = 0,\quad R^* = R+C$.

Lemma 6. $G \subset H$ *implies* $G^* \subset H^*$.

Lemma 7. $G_1 \cdot G_2 \cdots G_m = 0$ *implies* $G_1^* G_2^* \cdots G_m^* = 0$.

Proof. If $x \in G_i^*$, $i = 1, 2, \cdots, m$, then we have $x \in C$ and for any Cauchy family $\{X_\lambda;\ \lambda \in \Lambda\}$ of the class x there exist $\lambda_i \in \Lambda$ and $\alpha_i \in \Omega$ such that $S(X_{\lambda_i}, \mathfrak{U}_{\alpha i}) \subset G_i$, $i = 1, 2, \cdots, m$, and hence $G_1 G_2 \cdots G_m \supset X_{\lambda_1} X_{\lambda_2} \cdots X_{\lambda_m} \neq 0$, which contradicts the hypothesis of the lemma.

Now we take the set of G^* for all open sets G of R as a basis of open sets of R^*. Then R^* is clearly a space (in the sense of § 1) and R is a subspace of R^*.

Lemma 8. $\mathfrak{U}_\alpha^* = \{U^*;\ U \in \mathfrak{U}_\alpha\}$ *is an open covering of* R^*.

Proof. Let $x \in C$. For any $\alpha \in \Omega$ and any Cauchy family $\{X_\lambda\}$ of the class x there exist $X_\lambda \in \{X_\lambda\}$, $\beta \in \Omega$ and $U_\alpha \in \mathfrak{U}_\alpha$ such that $S(X_\lambda, \mathfrak{U}_\beta) \subset U_\alpha$, which shows that $x \in U_\alpha^*$.

Lemma 9. *If a point* x *of* $R^* - R$ *is contained in* G^*, *then we have* $S(x, \mathfrak{U}_\alpha^*) \subset G^*$ *for some* $\alpha \in \Omega$.

Proof. For a Cauchy family $\{X_\lambda\}$ of the class x there exist $X_\lambda \in \{X_\lambda\}$ and $\alpha \in \Omega$ such that $S(X_\lambda, \mathfrak{U}_\alpha) \subset G$. Let $x \in U_\alpha^*$, $y \in U_\alpha^*$ for some set U_α of $\mathfrak{U}_\alpha$. Then there exist $X_{\lambda_0} \in \{X_\lambda\}$ and $\beta \in \Omega$ such

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9) It is proved by the definition of equivalence that the condition holds for any $\{X_\lambda\}$ of the class x if it holds for some $\{Z_\nu\}$ of x.

that $S(X_{\lambda_0}, \mathfrak{U}_\beta) \subset U_\alpha$. If $y \in R$, then we have $y \in U_\alpha \subset S(X_\lambda, \mathfrak{U}_\alpha) \subset G$. If $y \in C$ and a Cauchy family $\{Y_\mu\}$ belongs to the class y, then there exist $Y_\mu \in \{Y_\mu\}$ and $\gamma \in \Omega$ such that $S(Y_\mu, \mathfrak{U}_\gamma) \subset U_\alpha$. Hence we have $S(Y_\mu, \mathfrak{U}_\gamma) \subset U_\alpha \subset S(X_\lambda, \mathfrak{U}_\alpha) \subset G$, that is, $y \in G^*$. Therefore $S(x, \mathfrak{U}_\alpha^*) \subset G^*$.

Lemma 10. *If $x \in R^* - R$, then we have*

$$x = [\prod_\alpha S(x, \mathfrak{U}_\alpha^*)] (R^* - R).$$

Proof. Let $y \in [\prod_\alpha S(x, \mathfrak{U}_\alpha^*)] (R^* - R)$. Then for any α there exists a set U_α of $\mathfrak{U}_\alpha$ such that $x, y \in U_\alpha^*$. By the argument in the proof of Lemma 9 we see that for a Cauchy family $\{Y_\mu\}$ of the class y there exist $Y_\mu \in \{Y_\mu\}$ and $\gamma \in \Omega$ such that $S(Y_\mu, \mathfrak{U}_\gamma) \subset S(X_\lambda, \mathfrak{U}_\alpha)$ for any $X_\lambda \in \{X_\lambda\}$. This shows that $\{X_\lambda\} \sim \{Y_\mu\}$.

Lemma 11. *If a vanishing Cauchy family $\{X_\lambda\}$ belongs to the class x which is a point of $R^* - R$, then we have $x = \prod_\lambda \bar{X}_\lambda$, where the bar indicates the closure operation in the space R^*.*

Proof. For any $\alpha \in \Omega$ there exist $X_{\lambda_0} \in \{X_\lambda\}$, $\beta \in \Omega$ and $U_\alpha \in \mathfrak{U}_\alpha$ such that $S(X_{\lambda_0}, \mathfrak{U}_\beta) \subset U_\alpha$. Hence we have $X_\lambda \cdot S(x, \mathfrak{U}_\alpha^*) \neq 0$, since $X_\lambda \cdot U_\alpha \neq 0$, $x \in U_\alpha^*$, and consequently $x \in \prod_\lambda \bar{X}_\lambda$ by Lemma 9. On the other hand, from the relation $S(X_{\lambda_0}, \mathfrak{U}_\beta) \subset U_\alpha$ it follows that $S(X_{\lambda_0}, \mathfrak{U}_\beta^*) \subset U_\alpha^*$. Hence we have $\bar{X}_{\lambda_0} \subset U_\alpha^* \subset S(x, \mathfrak{U}_\alpha^*)$. Therefore $\prod_\lambda \bar{X}_\lambda \subset \prod_\alpha S(x, \mathfrak{U}_\alpha^*)$. Since $\{X_\lambda\}$ is vanishing, we have $x = \prod_\lambda \bar{X}_\lambda$ by Lemma 10.

Lemma 12. *If G is an open set of R, then $S(G^*, \mathfrak{U}_\alpha^*) \subset [S(G, \mathfrak{U}_\alpha)]^*$.*

This Lemma follows immediately from Lemmas 6 and 7. Summarizing above results we obtain

Theorem 2. *R^* is a space which contains R as a subspace. R is dense in R^*, and every point of $R^* - R$ is closed.*

Theorem 3. *$\{\mathfrak{U}_\alpha^*\}$ is a uniformity of R^*. $\{\mathfrak{U}_\alpha^*\}$ is a T-uniformity, a regular uniformity or a completely regular uniformity, according as $\{\mathfrak{U}_\alpha\}$ is a T-uniformity, a regular uniformity or a completely regular uniformity.*

Theorem 4. *If a uniformity $\{\mathfrak{U}_\alpha\}$ of R agrees with the topology, then the uniformity $\{\mathfrak{U}_\alpha^*\}$ of R^* agrees with the topology.*

Proof. Let $x \in R$. If $S(x, \mathfrak{U}_\alpha) \subset G$, we have $S(x, \mathfrak{U}_\alpha^*) \subset G^*$.

We call R^* *the simple extension of R with respect to the uniformity $\{\mathfrak{U}_\alpha\}$.*

Remark. If $\{U; U \in \mathfrak{U}_\alpha, \alpha \in \Omega\}$ is a basis of open sets of R, then $\{U^*; U \in \mathfrak{U}_\alpha, \alpha \in \Omega\}$ is a basis of open sets of R^*.

§ 4. Further properties of R^*.

Lemma 13. *If $\{S(x, \mathfrak{U}_\alpha); \alpha \in \Omega\}$ is a basis of neighbourhoods of a point x of R, then we have*

(4) $$\prod_{\alpha} S(x, \mathfrak{U}_\alpha^*) = \prod_{\alpha} S(x, \mathfrak{U}_\alpha).$$

Proof. If $y \in (R^*-R) \cdot \prod_{\alpha} S(x, \mathfrak{U}_\alpha^*)$, then there exists, for any $\alpha \in \Omega$, a set U_α of $\mathfrak{U}_\alpha$ such that $x \in U_\alpha$ and $y \in U_\alpha^*$. For a Cauchy family $\{Y_\mu\}$ of the class y there exist $Y_{\mu_0} \in \{Y_\mu\}$ and $\beta \in \Omega$ such that $S(Y_{\mu_0}, \mathfrak{U}_\beta) \subset U_\alpha$. Hence we have $x \in U_\alpha \subset S(Y_\mu, \mathfrak{U}_\alpha)$ for every $Y_\mu \in \{Y_\mu\}$, and consequently we have $S(x, \mathfrak{U}_\alpha) \cdot Y_\mu \neq 0$, which shows that $x \in \prod_{\mu} \bar{Y}_\mu \cdot R$ by the hypothesis of the lemma. This contradicts the assumption that $\{Y_\mu\}$ is vanishing. Therefore $\prod_{\alpha} S(x, \mathfrak{U}_\alpha^*) \subset R$. This proves (4).

Lemma 14. *If $\{\mathfrak{U}_\alpha\}$ agrees with the topology, then*

(5) $$\prod S(x, \mathfrak{U}_\alpha^*) = x \text{ or } \bar{x} \cdot R,$$

according as $x \in R^*-R$ or $x \in R$.

Proof. Since $\prod S(x, \mathfrak{U}_\alpha^*) = \bar{x}$ by Theorem 4 and Lemma 1, we have (5) by Lemmas 11 and 13.

Theorem 5. *If R is a T-space and $\{\mathfrak{U}_\alpha\}$ is a T-uniformity of R, then R^* is a T-space. Furthermore, if R is a T_0-space, so is R^*.*

The first part of the theorem follows from the next Lemma 15. The second part is obvious.

Lemma 15. *If $\{\mathfrak{U}_\alpha; \alpha \in \Omega\}$ is a T-uniformity of a T-space R, then we have $(G_1 \cdot G_2)^* = G_1^* \cdot G_2^*$ for any open sets G_1, G_2 of R.*

Proof. Let $x \in G_1^* \cdot G_2^*$ and $x \in C$. Then for a Cauchy family $\{X_\lambda\}$ of the class x there exist $X_{\lambda_i} \in \{X_\lambda\}$ and $\alpha_i \in \Omega$ such that $S(X_{\lambda_i}, \mathfrak{U}_{\alpha_i}) \subset G_i$, $i = 1, 2$. If we take a common refinement $\mathfrak{U}_\beta$ of $\mathfrak{U}_{\alpha_1}$ and $\mathfrak{U}_{\alpha_2}$, then we have $S(X_{\lambda_1} \cdot X_{\lambda_2}, \mathfrak{U}_\beta) \subset G_1 G_2$. Let $S(X_\nu, \mathfrak{U}_\gamma) \subset U_\beta$ for some $X_\nu \in \{X_\lambda\}$, $\gamma \in \Omega$, $U_\beta \in \mathfrak{U}_\beta$. Then we have $S(X_\nu, \mathfrak{U}_\gamma) \subset S(X_{\lambda_1} X_{\lambda_2}, \mathfrak{U}_\beta) \subset G_1 G_2$. This proves Lemma 15.

Theorem 6. *If R is a T_1-space and $\{\mathfrak{U}_\alpha\}$ is a T-uniformity which agrees with the topology, then R^* is a T_1-space.*

Theorem 6 is a direct consequence of Theorem 5 and Lemmas 13, 14. The following theorem is also clear.

Theorem 7. *If R is a (completely) regular space and $\{\mathfrak{U}_\alpha\}$ is a (completely) regular uniformity which agrees with the topology, then R^* is a (completely) regu ar space.*

§ 5. Completeness. The case of regular uniformity.[10] A space R with a uniformity $\{\mathfrak{U}_\alpha\}$ is said to be *complete* with respect to the uniformity, if every Cauchy family $\{X_\lambda\}$ with respect to $\{\mathfrak{U}_\alpha\}$ is not vanishing, that is, $\prod_{\lambda} \bar{X}_\lambda \neq 0$.

Theorem 8. *A space R is complete with respect to the uniformyit $\{\mathfrak{U}_\alpha; \alpha \in \Omega\}$ which is composed of all open coverings of R.*

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10) The general case will be treated in the third note.

Proof. If a Cauchy family $\{X_\lambda ; \lambda \in \Lambda\}$ is vanishing, then $\{R-\bar{X}_\lambda ; \lambda \in \Lambda\}$ is an open covering of R, and hence it is equal to some $\mathfrak{U}_\alpha$. Since $\{X_\lambda\}$ is a Cauchy family there exist $\lambda \in \Lambda$ and $U_\alpha \in \mathfrak{U}_\alpha$ such that $X_\lambda \subset U_\alpha$. On the other hand, U_α is expressed as $R-\bar{X}_\mu$ with some $\mu \in \Lambda$. Hence we have $X_\lambda \cdot X_\mu = 0$, contrary to the finite intersection property.

Corollary. *A regular (or fully normal) space R is complete with respect to some regular (or completely regular) uniformity.*[11]

The extension R^* is not always complete, as will be shown below. Here we shall prove

Theorem 9. *If $\{\mathfrak{U}_\alpha\}$ is a (completely) regular uniformity of a space R which agrees with the topology, then R^* is complete with respect to the uniformity $\{\mathfrak{U}_\alpha^*\}$.*

Weil's theorem and Cohen's theorem are contained in our Theorem 9.[12] We first prove some lemmas.

Lemma 16. *Let $\{\mathfrak{U}_\alpha ; \alpha \in \Omega\}$ be a regular uniformity of a space R. Then a family $\{X_\lambda\}$ of subsets of R with the finite intersection property is a Cauchy family if for any $\alpha \in \Omega$ there exist a set $X_\lambda \in \{X_\lambda\}$ and a set U_α of $\mathfrak{U}_\alpha$ such that $X_\lambda \subset U_\alpha$.*

Lemma 17. *Let $\{X_\lambda\}$ and $\{Y_\mu\}$ be Cauchy families with respect to a regular uniformity $\{\mathfrak{U}_\alpha ; \alpha \in \Omega\}$. Then $\{X_\lambda\} \sim \{Y_\mu\}$, if for any $\alpha \in \Omega$ and any $X_\lambda \in \{X_\lambda\}$ there exists a set $Y_\mu \in \{Y_\mu\}$ such that $Y_\mu \subset S(X_\lambda, \mathfrak{U}_\alpha)$.*

Since Lemma 16 is clear, we have only to prove Lemma 17. For any $\alpha \in \Omega$ there exist $X_{\lambda_0} \in \{X_\lambda\}$, $\beta \in \Omega$ and $U_{\lambda(\alpha)} \in \mathfrak{U}_{\lambda(\alpha)}$ such that $S(X_{\lambda_0}, \mathfrak{U}_\beta) \subset U_{\lambda(\alpha)}$. Let $Y_\mu \subset S(X_{\lambda_0}, \mathfrak{U}_\beta)$. Then we have $S(Y_\mu, \mathfrak{U}_\delta) \subset S(X_\lambda, \mathfrak{U}_\alpha)$, where $\delta = \delta(\alpha. U_{\lambda(\alpha)})$.

Cororally. $\{X_\lambda\} \sim \{Y_\mu\}$ *if and only if* $\{X_\lambda + Y_\mu\}$ *is a Cauchy family. Here $\{\mathfrak{U}_\alpha\}$ is assumed to be a regular uniformity (or a T-uniformity).*

Proof of Theorem 9. Let $\{M_\lambda ; \lambda \in \Lambda\}$ be a Cauchy family of R^* with respect to $\{\mathfrak{U}_\alpha^* ; \alpha \in \Omega\}$. According to Lemmas 16 and 17 $\{S(M_\lambda, \mathfrak{U}_\alpha^*) ; \lambda \in \Lambda, \alpha \in \Omega\}$ is a Cauchy family which is equivalent to $\{M_\lambda\}$. By Lemma 7 $\{R \cdot S(M_\lambda, \mathfrak{U}_\alpha^*)\}$ is a Cauchy family of R with respect to $\{\mathfrak{U}_\alpha\}$. Hence we have $\Pi \overline{S(M_\lambda, \mathfrak{U}_\alpha^*) \cdot R} \neq 0$, and consequently $\Pi \bar{M}_\lambda \neq 0$ by Lemma 4. Thus R^* is complete.

Example. In case $\{\mathfrak{U}_\alpha\}$ is a completely regular uniformity which does not agree with the topology, R^* is not neccessarily complete even if R is a metrizable space. Let R be a subspace

11) This is proved for metric spaces by J. Dieudonne (Ann. L'ecole norm. sup. 56 (1939), p. 280) and for fully normal spaces by T. Shirota (Shijo-Danwakai, 9 (1948), p. 283), and by the present author (ibid., 13 (1949), p. 458).

12) Cf. footnotes 1), 2) and the remark at the end of § 1.

of a two-dimensional Euclidean space such that $R=\{(x, y);\ 0<x<1,\ 0<y<1\}+\{(x, 0);\ 0\leqq x\leqq 1\}+\{(x, 1);\ 0\leqq x\leqq 1\}$. Let us denote by U_{nj} the intersection of the set $\{(x, y);\ 0\leqq x\leqq 1,\ \frac{j-1}{3^n}<y<\frac{j+1}{3^n}\}$ with R and put $\mathfrak{U}_n=\{U_{nj};\ j=0, 1,\cdots, 3^n\}$. Then it is easy to see that $\{\mathfrak{U}_n\}$ is a completely regular uniformity of R. For any real number α a Cauchy family $\left\{\sum_{i=m}^{\infty}\left(\frac{1}{i+1}, \alpha\right);\ m=1, 2,\cdots\right\}$ defines a point of R^* which will be denoted by $p^*(\alpha)$. Then $R^*=R+\{p^*(\alpha);\ 0<\alpha<1\}$, and $\left\{\sum_{i=m}^{\infty}p^*\left(\frac{1}{i+1}\right);\ m=1, 2,\cdots\right\}$ is a vanishing Cauchy family with respect to $\{\mathfrak{U}_n^*\}$. Thus R^* is not complete, (while R^{**} is complete).