39. *Weak Topology and Compact Open Topology.*

By Hisaharu UMEGAKI.

Mathematical Institute, Kyûsyû University.

(Comm. by K. KUNUGI, M.J.A., April 12, 1951.)

In a recent paper, J. Dieudonné [1][1] has proved the following Theorem concerning the relation between weak topology and compact open topology (which is called k-topology by R. Arens [2]).

Theorem (Dieudonné). *The bounded weak* topology in a Banach space is identical with the compact open topology.*

On the other hand we have given a relation between weak topology and compact open topology in the following general form (cf. [3]):

Let X be a T_1-topological space, Y a uniform space defined by the uniform structure $[V_\alpha]$ and further let C be an arbitrary family of continuous transformations from X to Y. C is called equi-continuous (p) when, for any $p \in X$ and for any V_α, there exists a n.b.d. $V(p)$ of p such that $q \in V(p) \to f(q) \in V_\alpha(f(p))$ for all $f \in C$. Moreover, we induce two structures in C as follows:

$$W_{\alpha, p_1, \dots, p_n} = \{(f, g)\,;\ (f(p_i), g(p_i)) \in V_\alpha,\ i = 1, 2, \dots, n\} \quad (p_i \in X)$$

and

$$W_{\alpha, K} = \{(f, g)\,;\ (f(p), g(p)) \in V_\alpha,\ p \in K\}$$

K being a compact set of X. We shall call these topologies weak and compact open, respectively, and denote them by τ_w and τ_k.

Theorem 1. *If C is equi-continuous (p), then weak topology is identical with compact open topology.*

Proof. After R. Arens [3], τ_k is weaker than any admissible topology in C. Then, it is sufficient to prove that τ_w is admissible. For any $p_0 \in X$ and $f_0 \in C$, $W_{\alpha, p_0}(f_0)$ be τ_w-n.b.d. in C. Since C is equi-continuous, there exists a n.b.d. $V(p_0)$ of p_0 such that $q \in V(p_0) \to f(q) \in V_\alpha(f(p_0))$ for all $f \in C$. Thus, for $f \in W_{\alpha, p_0}(f_0)$ and $p \in V(p_0)$, we have $f(p) \in V_\alpha(f_0(p_0))$, and then $f(p)$ is continuous in the product topology (τ_w, X).

By this Theorem, we may easily prove the Dieudonné's theorem. Let E and E^* be Banach space and its conjugate space respectively, and define the weak*-n.b.d. and k^*-n.b.d. as follows:

$$U(f_0, \varepsilon, x_1, \dots x_n) = \{f;\ |f_0(x_i) - f(x_i)| < \varepsilon,\ i = 1, 2, \dots, n\}, \quad x_i \in E$$

1) Number in References at the end of this paper.

and

$$U(f_0, \varepsilon, K) = \{f;\ |f_0(x)-f(x)| < \varepsilon,\ x \in K\}$$

K being a strongly compact set of E. The compact open topology in J. Dieudonné's Theorem is defined by the above k^*-n.b.d. Let the unit sphere of E^* be S^*, then S^* is equi-continuous and by our theorem the two topologies defined by weak* n.b.d. and k^* n.b.d. are identical. Thus J. Dieudonne's Theorem follows from our Theorem.

Using bounded weak topology instead of the bounded weak* topology, we can prove the following theorem :

Theorem 2. *In a Banach space E, bounded weak topology and compact open topology are identical, where the compact open topology is defined by the k-n.b.d. :*

$$U(x_0, \varepsilon, K^*) = \{x;\ |f(x_0)-f(x)| < \varepsilon, f \in K^*\},$$

K^* being strongly compact in E^*.

Finally we remark that the identity of convergences of positive definite functions on L.C. group has been introduced for weak topology and compact open topology (e.g. [4]). But since, on this case that compact open topology in different from ours we shall not discuss of this paper. It will be apper the next paper.

References.

1. J. Dieudonné: Natural Homomorphisms in Banach Space. Proc. of the Amer. Math. Soc., vol. 1. No. 1 (1950).

2. R. Arens: A topology for space of transformations. Annals of Mathematics 47 (1946).

3. H. Umegaki: Compact set in uniform space and functions space. Tôhoku Math. J. vol. 3. No. 1 (in the press).

4. H. Yoshizawa: On some type of convergence of positive definite functions. Osaka Math. J., vol. 1. No. 1 (1949).