74. On Some Representation Theorems in an Operator Algebra. I.

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I. E. Segal has proved that a state of a C^*-algebra is the normalizing function of some normal representation (cf. [2][1]). (C^*-algebra is a uniformly closed self-adjoint operator algebra on a Hilbert Space, in the terminology of I. E. Segal.) Applying the Reduction Theory of J. von Neumann (cf. [1]) for this theorem, we can see that a state of a separable C^*-algebra is a directed integral of a system of pure states, and we can see a similar result for trace instead of state (the terminologies of state, pure state and trace etc. are of I. E. Segal [2] and M. Nakamura [3] respectively). This is a Theorem of Bochner's type in $*$-algebra. From this, we can easily see by a topological method the Bochner's Theorem for a separable locally compact group. (Recently, this theorem has been shown by F. I. Mautner [6].)

M. Nakamura [3] has introduced the two-sided representation of a C^*-algebra which is a generalized form of double unitary representation in the sense of R. Godement [8]. From his formulation, we can see that a two sided representation of a C^*-algebra is a directed integral of a system of irreducible two-sided representations. From this fact and the Bochner's Theorem, any two-sided continuous unitary representation in a separable unimodular locally compact group is a directed integral of a system of irreducible two sided continuous unitary representations, it follows the same type theorem of F. I. Mautner [6] for one-sided continuous unitary representation of the group.

We shall describe in this paper only on a weight function $\sigma(\lambda)$ which generates the irreducible factors. But it may be possible to prove a decomposition for any N-function in the sense of von Neumann (cf. [1]) as a weight function.

Throughout this paper, we shall assume the separability axiom, because we shall use the Reduction Theory of J. von Neumann.

1. A Bochner's type Theorem in a C^*-algebra. Recently, the Theorem of this type has been proved for the case of non-separable central C^*-algebra by M. Nakamura—Y. Misonou [4], and for the

1) Number in Bibliography at the end of this paper.

case of commutative *-algebra by R. Godement [9]. In this section, we shall prove this theorem for any separable C^*-algebra.

Let $\mathfrak{A}$ be a separable C^*-algebra. A *state*[1] of $\mathfrak{A}$ is a complex-valued bounded linear functional on $\mathfrak{A}$ such that $\omega(x^*)$ is the complex conjugate of $\omega(x)$, $\omega(x^*x) \geqq 0$ for all $x \in \mathfrak{A}$ and $\sup \omega(x^*x) = \|\omega\| (= \sup |\omega(x)|)$. A state $\tau(x)$ is a *trace* if $\tau(xy) = \tau(yx)$ for all $x, y \in \mathfrak{A}$. A state (resp. trace) $\chi(x)$ is *pure* if it is not a linear combination with positive coefficients of two other states (resp. traces). A C^*-algebra or a L-algebra of a locally compact group has sufficiently many pure states (cf. [2]). On the case of the trace, it has been discussed in a central C^*-algebra by M. Nakamura—Y. Misonou [4] and in a central group by R. Godement [10].

Theorem 1. *Let $\omega(x)$ be a state (resp. trace) on $\mathfrak{A}$. Then*

$$\omega(x) = \int_R \chi(x, \lambda) d\sigma(\lambda) \tag{1}$$

where $\sigma(\lambda)$ is a suitable bounded real valued non-decreasing right continuous function on real line R which is a N-function of the sense of J. von Neumann [1], *and $\chi(x, \lambda)$ is a pure state on $\mathfrak{A}$ for almost every λ in R with respect to $\sigma(\lambda)$-measure.*

Proof. Let $\mathfrak{K}$ be a set of $x \in \mathfrak{A}$ such that $\omega(yx) = 0$ for all $y \in \mathfrak{A}$, then $\mathfrak{K}$ is a closed left-ideal in $\mathfrak{A}$. Hence we can make the factor space $[\mathfrak{A}] = \mathfrak{A}/\mathfrak{K}$, we shall denote by $[x]$ the class containing x. Define

$$([y], [x]) = \omega(x^* y) \tag{2}$$

for x, y of $\mathfrak{A}$, then (2) is an inner product in $[\mathfrak{A}]$. Let $\mathfrak{H}$ be the completion of $[\mathfrak{A}]$ by the norm $\|[x]\|^2 = ([x], [x])$. Then $\mathfrak{H}$ is a separable Hilbert space, and the mapping from x of $\mathfrak{A}$ to $[x]$ of $\mathfrak{H}$ is continuous. We shall define a representation of $\mathfrak{A}$ on $\mathfrak{H}$ by the following way:

$$x \rightarrow U_x: \qquad U_x[y] = [xy].$$

Then, by the Theorem of I. Segal [2], the state $\omega(x)$ be represented by

$$\omega(x) = (U_x\xi, \xi) \tag{3}$$

for some element $\xi \in \mathfrak{H}$. Let $\boldsymbol{A}$ be a maximal commutative self-adjoint subalgebra of $\boldsymbol{M}'$ which is the commutor of $\boldsymbol{M} = \{U_x \mid x \in \mathfrak{A}\}$. The decomposition of $\mathfrak{H}$ and $\boldsymbol{M}$ with respect to $\boldsymbol{A}$ be

1) It can be seen that every positive bounded linear functional on $\mathfrak{A}$ is a state in our sense, because it satisfies the Schwarz' inequality and $\mathfrak{A}$ has an approximate identity.

$$\mathfrak{H} = \int_R \mathfrak{H}_\lambda \sqrt{d\sigma(\lambda)}, \qquad \boldsymbol{M} \sim \sum \boldsymbol{M}(\lambda)$$

where $\sigma(\lambda)$ is a weight function generated by $\boldsymbol{A}$. By the theorem of F. Mautner [7], almost every $\mathfrak{H}_\lambda$ is irreducible under $\boldsymbol{M}(\lambda)$. We denote $\xi = \int_R \xi_\lambda \sqrt{d\sigma(\lambda)}$ and $U_x \sim \sum U_x(\lambda)$ for $\xi \in \mathfrak{H}$ and $x \in \mathfrak{A}$. Then we have

$$(4) \qquad (U_x\xi, \xi) = \int_R (U_x(\lambda)\xi_\lambda, \xi_\lambda) d\sigma(\lambda),$$

and put $\chi(x, \lambda) = (U_x(\lambda)\xi_\lambda, \xi_\lambda)$ a.e. $\sigma(\lambda)$-measure. By the Reduction Theory of von Neumann, for the decomposition

$$U_x \sim \sum U_x(\lambda) \qquad \text{and} \qquad U_y \sim \sum U_y(\lambda)$$

we have

$$U_{xy} = U_x U_y \sim \sum U_{xy}(\lambda), \quad \sim \sum U_x(\lambda) U_y(\lambda),$$
$$(5) \qquad U_{x^*} = U_{x^*} \sim \sum U_{x^*}(\lambda), \quad \sim \sum U_{x^*}(\lambda)$$

and the decomposition is unique for all $x, y \in \mathfrak{A}$ except for a set of $\sigma(\lambda)$-measure zero, since $\mathfrak{A}$ is separable. It follow that almost all λ $\{U_x(\lambda), \mathfrak{H}_\lambda\}$ is an irreducible representation of $\mathfrak{A}$, and $\chi(x, \lambda)$ is a normalizing function of $\{U_x(\lambda), \mathfrak{H}_\lambda\}$. Hence almost all λ $\chi(x, \lambda)$ is a pure state. Thus we have the relation (1) for the case of state.

Remark 1. Theorem 1 can be also hold for a case that a complete normed*-algebra with an approximate identity and a state. For, in such a *-algebra.

$$\omega(y^*x^*xy) \leq \lim \|(x^*x)^n\|^{1/n} \omega(y^*y) \leq \|x\|^2 \cdot \omega(y^*y)$$

and therefore $\|U_x[y]\| \leq \|x\| \, \|[y]\|$, or $|||U_x||| \leq \|x\|$ where $|||\cdot|||$ is the operator norm. Since a.e. $\sigma(\lambda)$ $|||U_x(\lambda)||| \leq |||U_x|||$, a.e. $\sigma(\lambda)$ the representations $\{U_x(\lambda), \mathfrak{H}_\lambda\}$ are continuous. (It is known that any representation of a B^*-algebra is necessarily continuous, and from above fact it also hold in our case.) It can be seen by the same way on the case of C^*-algebra that a state is a normalizing function of the corresponding representation and conversely a normalizing function of a normal representation is a state. Thus, Theorem 1 be held for any such an algebra. This fact will be used for the proof of the Bochner's Theorem in a topological group (Theorem 3, below).

2. On a decomposition of a two-sided representation of a C^*-algebra. The two-sided representation of a C^*-algebra has been introduced by M. Nakamura [3], it is a general case for a locally compact group introduced by R. Godement (cf. [8]) which he has called double unitary representation.

An involution j is a conjugate linear transformation on a Hilbert space $\mathfrak{H}$ of preriod two onto itself with $(j\xi, j\eta) = (\eta, \xi)$ for any $\xi, \eta \in \mathfrak{H}$. $\{U_x, V_x, j, \mathfrak{H}\}$ is a *two-sided representation* of a C^*-algebra $\mathfrak{A}$, if $x \to U_x$ is a usual representation and $x \to V_x$ is a dual representation:

$$(6) \qquad V_{xy} = V_y V_x, \quad U_x V_y = V_y U_x \quad \text{and} \quad V_x^* = jU_x j.$$

M. Nakamura has proved that a C^*-algebra with a trace has a normal two-sided representation. We have

Theorem 2. *A normal two-sided representation of a reparable C^*-algebra is a directed integral of a system of irreducible two-sided representations*[1].

Proof. Let $\{U_x, V_x, j, \mathfrak{H}\}$ be a two-sided representation, and $\boldsymbol{A}$ be a maximal commutative self-adjoint subalgebra of $\boldsymbol{M}'$ such that every element A of $\boldsymbol{A}$ satisfying $jAj = A^*$, where $\boldsymbol{M}'$ is a commutor of $\boldsymbol{M} = \{U_x, V_y \mid x, y \in \mathfrak{A}\}$. By the same way in Theorem 1, decompose $\mathfrak{H}$, U_x and V_x with respect to $\boldsymbol{A}$:

$$\mathfrak{H} = \int \mathfrak{H}_\lambda \sqrt{d\sigma(\lambda)}, \quad U_x \sim \sum U_x(\lambda) \quad \text{and} \quad V_x \sim \sum V_x(\lambda).$$

$$V_{xy} = V_y V_x \sim \sum V_{xy}(\lambda), \quad \sim \sum V_y(\lambda) V_x(\lambda).$$

$$U_x V_y = V_y U_x \sim \sum V_x(\lambda) V_y(\lambda), \quad \sim \sum V_y(\lambda) U_x(\lambda).$$

$$V_{x^*} = V_x^* \sim \sum V_{x^*}(\lambda), \quad \sim \sum V_x^*(\lambda).$$

and the decomposition is unique (a.e. $\sigma(\lambda)$) for all $x, y \in \mathfrak{A}$ since $\mathfrak{A}$ is separable, and therefore

$$V_{xy}(\lambda) = V_y(\lambda) V_x(\lambda), \qquad V_{x^*}(\lambda) = V_x^*(\lambda),$$
$$U_x(\lambda) V_y(\lambda) = V_y(\lambda) U_x(\lambda)$$

for all $x, y \in \mathfrak{A}$ (a.e. $\sigma(\lambda)$). We have already proved in Theorem 1 that $\{U_x(\lambda), \mathfrak{H}_\lambda\}$ is a usual representation (a.e. $\sigma(\lambda)$). Now we shall research $j(\lambda)$ which is a component on $\mathfrak{H}_\lambda$ of the decomposition of j, and prove that $j(\lambda)$ is our involution on $\mathfrak{H}_\lambda$ (a.e. $\sigma(\lambda)$). For any $\xi \in \mathfrak{H}$, denote $\xi = \int \xi_\lambda \sqrt{d\sigma(\lambda)}$ and $j\xi = \int \zeta_\lambda \sqrt{d\sigma(\lambda)}$ and define $j(\lambda)$ such as

$$j(\lambda): \quad \zeta_\lambda = j(\lambda)\xi_\lambda, \qquad \text{a.e. } \sigma(\lambda).$$

The decomposition $j \sim \sum j(\lambda)$ be possible becouse $A^* = jAj$ for all $A \in \boldsymbol{A}$.

Since $(j\xi, j\eta) = (\eta, \xi)$ for arbitrary $\xi, \eta \in \mathfrak{H}$,

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1) A two-sided representation $\{U_x, V_x, j, \mathfrak{H}\}$ is irreducible if no proper subspace of $\mathfrak{H}$ exists which is invariant under U_x, V_x $(x \in \mathfrak{A})$ and j (cf. [3])

$$\int_R (j(\lambda)\xi_\lambda, j(\lambda)\eta_\lambda) d\sigma(\lambda) = \int_R (\eta_\lambda, \xi_\lambda) d\sigma(\lambda)$$

and $(j\xi, jB\eta) = (B\eta, \varepsilon)$ for any bounded operator B on $\mathfrak{H}$.
Let $\varphi(\lambda)$ be a bounded real valued $\sigma(\lambda)$-measurable function on R and we define a bounded operator B_φ:

$$B_\varphi = \int_R \varphi(\lambda)\eta_\lambda^{\cdot}\sqrt{d\sigma(\lambda)} \tag{7}$$

where

$$\eta = \int \eta_\lambda \sqrt{d\sigma(\lambda)}, \quad \text{then } B_\varphi \text{ is a bounded operator on } \mathfrak{H}.$$

Therefore, for any bounded real valued $\sigma(\lambda)$-measurable function $\varphi(\lambda)$

$$\int_R \varphi(\lambda)(j(\lambda)\xi_\lambda, \ j(\lambda)\eta_\lambda) d\sigma(\lambda) = \int_R \varphi(\lambda)(\eta_\lambda, \xi_\lambda) d\sigma(\lambda). \tag{8}$$

However, any bounded complex $\sigma(\lambda)$-measurable function $\phi(\lambda)$ is decomposed into φ_1 and φ_2 (real) such that $\phi(\lambda) = \varphi_1(\lambda) + i\varphi_2(\lambda)$, and therefore (8) holds for any such function $\phi(\lambda)$ on R. Hence we obtain

$$(j(\lambda)\xi_\lambda, j(\lambda)\eta_\lambda) = (\eta_\lambda, \xi_\lambda), \quad \text{a.e. } \sigma(\lambda).$$

Since $\mathfrak{A}$ is separable, for all $x \in \mathfrak{A}$

$$j(\lambda)U_x(\lambda)j(\lambda) = V_x^*(\lambda), \quad \text{a.e. } \sigma(\lambda).$$

By Mautner's Theorem, almost all λ, $\mathfrak{H}_\lambda$ are irreducible under $\boldsymbol{M}(\lambda)$. Hence a.e. $\sigma(\lambda)$ $\{U_x(\lambda), V_x(\lambda), j(\lambda), \mathfrak{H}_\lambda\}$ are irreducible two-sided representations and its directed integral with respect to $\sigma(\lambda)$-measure is $\{U_x, V_x, j, \mathfrak{H}\}$.

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