130. *Probability-theoretic Investigations on Inheritance. IV$_3$. Mother-Child Combinations.*

(Further Continuation.)

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3. Mother-children combination concerning families with two childeren.

We have hitherto been interested in combinations consisting of a mother with her one child. Now, we can treat similar problem on those of a mother with her two children. We shall call two children, the order of which is also taken into account, briefly the first child and the second child. But, they may be, in general, any two childeren chosen among their brethren or sisters; it will be noticed that they are not necessarily the first-born and the second-born in the strict sense. In this two-children case, various new interesting results will be derived in comparison with the previous one-child case. Moreover, the results on the latter case are contained in the present case as special ones. *Two children belonging to the same family mean, in the following, those having both parents in common.* Now, both children belonging to the same family, or more generally those having mother alone in common will possess the types which are not quite independent each other but between which certain correlation is existent. In fact, genotype of each child must then contain at least one gene in common with that of mother. If, in particular, mother is homozygotic, genotypes of both children contain always at least one gene, namely the one composing the type of mother, in common each other.

Now, we consider an inheritance character which consists of m genes $A_i (i=1,\dots,m)$ with distribution-probability $\{p_i\}$, the distribution being supposed to be in an equibrium state. The number of permutations, admitting the repetition, of selecting any two types of children without kinship is evidently equal to $\frac{1}{4}m^2(m+1)^2$. On the other hand, that of selecting any two children having a common mother is equal to m^2 or $(2m-1)^2$ according to the mother of homozygote or of heterozygote, respectively. If they are further restricted such as to have a father also in common, then, the number of possible permutations reduces to a small number. In fact, as seen from the table in § 3 of I, we get the following table.

Mating	$A_{ii}\times A_{ii}$	$A_{ii}\times A_{ik}$	$A_{ii}\times A_{hh}$	$A_{ii}\times A_{hk}$	$A_{ij}\times A_{ij}$	$A_{ij}\times A_{ik}$	$A_{ij}\times A_{hk}$
Permutation	1	4	1	4	9	16	16

The circumstances are similar also for phenotypes.

We shall generalize the table in §3 of I to two-children case. Now the order of members in every mating must be taken into account, as done in §1 of IV. For this purpose, it is only necessary to divide each mating between different types into two classes, being assigned to each class half value of corresponding probability.

Using the results on one-child case, the corresponding results on two-children case can immediately be written down. For example, the mating between mother A_{ij} and father A_{ij} produces a child A_{ii} or A_{ij} with probability 1/4 or 1/2, respectively, and hence a pair consisting of the first child A_{ii} and the second child A_{ii} with probability $1/4\times 1/4=1/16$, and a pair consisting of the first child A_{ii} and the second child A_{ij} with probability $1/4\times 1/2=1/8$.

We now construct a table on mother-children combination with two children. We denote by $\pi(A_{ij};\, A_{hk},\, A_{fg})$ or more briefly by

$$(3.1)\qquad \pi(ij\,;\,hk,fg)\qquad (i,j,h,k,f,g=1,\dots,m)$$

the probability of appearing of a combination of a mother A_{ij} and her first child A_{hk} and second child A_{fg}, produced from a common father. Based on a principle of Mendelian inheritance, this quantity vanishes provided either

$$(3.2)\quad (i-h)(i-k)(j-h)(j-h)\neq 0 \text{ or } (i-f)(i-g)(j-f)(j-g)\neq 0.$$

The quantity (3.1) is symmetric with respect to i and j, to h and k, and to f and g; namely, there exist identical relations

$$(3.3)\quad \begin{aligned}&\pi(ij\,;\,hk,fg)=\pi(ji\,;\,hk,fg)=\pi(ij\,;\,kh,fg)=\pi(ji\,;\,kh,fg)\\ &=\pi(ij\,;\,hk,gf)=\pi(ji\,;\,hk,gf)=\pi(ij\,;\,kh,gf)=\pi(ji\,;\,kh,gf)\end{aligned}$$

for any i,j,h,k,f,g. Consequently, we can and will make a similar *agreement*, as before, that the sum of eight quantities (3.3), eventually four or two or one if one or two or three homozygotes are related there, will anew be represented by any one among them. Based on this agreement, m^6 quantities (3.1) reduce to $(\frac{1}{2}m(m+1))^3=\frac{1}{8}m^3(m+1)^3$ in number. On the other hand, most of these quantities vanish identically in view of (3.9). There remain m^3 or $\frac{1}{2}m(m-1)(2m-1)^2$ non-vanishing quantities according to mothers of homozygote or of heterozygote; the sum is $\frac{1}{2}m(4m^3-6m^2+5m-1)$. Besides the trivial relations (3.3), there exist further symmetry relations with respect to both children; namely, we have

$$(3.4)\qquad \pi(ij\,;\,hk,fg)=\pi(ij\,;\,fg,hk)$$

identically for any set i,j,h,k,f,g. Hence, among the non-vanishing quantities, any one is equal to that obtained by interchanging the order of the children. Thus, *the whole number of non-vanishing quantities different each other reduces to at most* $\frac{1}{2}(\frac{1}{2}m(4m^3-6m^2+5m-1)+\frac{1}{2}m(2m^2-m+1))=m^2(m^2-m+1)$.

We first consider a mother of homozygote A_{ii}. Possible type of a father which can produce a pair of children (A_{ii}, A_{ii}) with her are those containing the gene A_i at least one, namely A_{ii}, A_{ih}, $A_{ik}(h, k \neq i; h \neq k)$. But, since the types A_{ih} and A_{ik} among them are characterized merely by possession of only one gene A_i in common with that of the mother, the type A_{ik} may be brought together into $A_{ih}(h \neq i)$. Thus, we get, by means of the above table,

$$(3.5)\quad \pi(ii;\, ii,\, ii) = 1p_i^4 + \tfrac{1}{4}2p_i^3 \sum_{h \neq i} p_h = p_i^4 + \tfrac{1}{2}p_i^3(1-p_i) = \tfrac{1}{2}p_i^3(1+p_i).$$

The type of a father who can produce a pair of children $(A_{ii}, A_{ih})(h \neq i)$ with a mother A_{ii} is uniquely determined as to be A_{ih}, and hence we obtain

$$(3.6)\qquad \pi(ii;\, ii,\, ih) = \tfrac{1}{4}2p_i^3p_h = \tfrac{1}{2}p_i^3p_h \qquad (h \neq i).$$

By means of the symmetry relation (3.4), we get, from (3.6),

$$(3.7)\qquad \pi(ii;\, ih,\, ii) = \tfrac{1}{2}p_i^3p_h \qquad (h \neq i).$$

Possible types of a father who can produce a pair of children $(A_{ih}, A_{ih})(h \neq i)$ with a mother A_{ii} are A_{ih}, A_{hh} and $A_{hk}(k \neq i, h)$. Hence, we obtain

$$(3.8)\qquad \begin{aligned} \pi(ii;\, ih,\, ih) &= \tfrac{1}{4}2p_i^2p_h + 1p_i^2p_h^2 + \tfrac{1}{4}2p_i^2p_h \sum_{k \neq i, h} p_k \\ &= \tfrac{1}{2}p_i^2p_h(1+p_h) \qquad\qquad (h \neq i). \end{aligned}$$

For combination of a mother A_{ii} with children (A_{ih}, A_{ik}) $(h, k \neq i;\, h \neq k)$, the unique type A_{hk} of a father is only taken into account, and hence we get

$$(3.9)\quad \pi(ii;\, ih,\, ik) = \tfrac{1}{4}2p_i^2p_hp_k = \tfrac{1}{2}p_i^2p_hp_k \qquad (h, k \neq i;\, h).$$

We next consider a mother of heterozygote $A_{ij}\,(i \neq j)$. Possible types of a father which can produce children (A_{ii}, A_{ii}) are then A_{ii}, A_{ij}, A_{ih} $(h \neq i, j)$, A_{ik} $(k \neq i, j, h)$, or more briefly expressed, A_{ii}, A_{ij}, A_{ih} $(h \neq i, j)$. Hence, by means of the table, we obtain

$$(3.10)\quad \pi(ij;\, ii,\, ii) = \tfrac{1}{4}2p_i^3p_j + \tfrac{1}{16}4p_i^2p_j^2 + \tfrac{1}{16}4p_i^2p_j \sum_{h \neq i, j} p_h = \tfrac{1}{4}p_i^2p_j(1+p_i).$$

We now get immediately

$$(3.11)\qquad \pi(ij;\, ii,\, jj) = \tfrac{1}{16}4p_i^2p_j^2 = \tfrac{1}{4}p_i^2p_j^2,$$

and, similarly to (3.10),

$$(3.12)\quad \begin{aligned} &\pi(ij;\, ii,\, ij) \\ &= \tfrac{1}{4}2p_i^3p_j + \tfrac{1}{8}4p_i^2p_j^2 + \tfrac{1}{16}4p_i^2p_j \sum_{h \neq i, j} p_h = \tfrac{1}{4}p_i^2p_j(1+p_i+p_j). \end{aligned}$$

The respective relations

$$(3.13)\quad \pi(ij;\, ii,\, ii) = \tfrac{1}{4}p_i^2p_jp_h,\quad \pi(ij;\, ii,\, ik) = \tfrac{1}{4}p_i^2p_jp_k \quad (h, k \neq i, j),$$

$$(3.14)\quad \pi(ij;\, ii,\, jh) = \tfrac{1}{4}p_i^2p_jp_h,\quad \pi(ij;\, ii,\, jk) = \tfrac{1}{4}p_i^2p_jp_k \quad (h, k \neq i, j)$$

are trivially equvialent within each pair. However, comparing the both pairs mutually, we conclude a remarkable relation

$$(3.15)\qquad \pi(ij;\, ii,\, ih) = \pi(ij;\, ii,\, jh) \quad (h \neq i, j)$$

yielding that, if a mother is of a heterozygote and the first child of a homozygote and if the second child has a gene different from the mother, then the probability of combination is independent of another gene of the second child.

We get further the relations equivalent to (3.11), (3.10), (3.12), (3.13), (3.14), respectively, by interchanging the suffices i and j.

The relation corresponding to (3.15) now becomes $\pi(ij; jj, ih) = \pi(ij; jj, jh)$ $(h \neq i, j)$ which is also essentially equivalent to (3.15).

The probability in which mother and the first child are of the same type $A_{ij}(i \neq j)$ is derived by means of corresponding part of the table. Thus, we get two relations equivalent each other, stating that

$$(3.16) \qquad \begin{aligned} &\pi(ij;\, ij,\, ii) \\ &= \tfrac{1}{4} 2p_i^3 p_j + \tfrac{1}{8} 4p_i^2 p_j^2 + \tfrac{1}{16} 4p_i^2 p_j \sum_{h \neq i,j} p_h = \tfrac{1}{4} p_i^2 p_j (1 + p_i + p_j), \end{aligned}$$

$$(3.17) \qquad \pi(ij;\, ij,\, jj) = \tfrac{1}{4} p_i p_j^2 (1 + p_i + p_j),$$

which may also be derived from (3.12), based on the symmetry relation (3.4). In case where mother and both children are of the same heterozygote $A_{ij}(i \neq j)$, we get

$$(3.18) \qquad \begin{aligned} &\pi(ij;\, j,\, ij) = \tfrac{1}{4} 2p_i^3 p_j + \tfrac{1}{4} 2p_i p_j^3 + \tfrac{1}{4} p_i^2 p_j^2 \\ &+ \tfrac{1}{16} 4p_i^2 p_j \sum_{h \neq i,j} p_h + \tfrac{1}{16} 4p_i p_j^2 \sum_{h \neq i,j} p_h = \tfrac{1}{4} p_i p_j (p_i + p_j)(1 + p_i + p_j). \end{aligned}$$

We obtain further

$$(3.19) \qquad \begin{aligned} \pi(ij;\, ij,\, ih) &= \tfrac{1}{4} p_i p_j p_h (p_i + p_j), \\ \pi(ij;\, ij,\, ik) &= \tfrac{1}{4} p_i p_j p_k (p_i + p_j) \end{aligned} \qquad (h,\, k \neq i,\, j),$$

$$(3.20) \qquad \begin{aligned} \pi(ij;\, ij,\, jh) &= \tfrac{1}{4} p_i p_j p_h (p_i + p_j), \\ \pi(ij;\, ij,\, jk) &= \tfrac{1}{4} p_i p_j p_k (p_i + p_j) \end{aligned} \qquad (h,\, k \neq i,\, j),$$

whence it follows, corresponding to (3.15), the equality $\pi(ij;\, ij,\, ih)$ $\pi(ij;\, ij,\, jh)$ $(h \neq i, j)$.

In quite a similar manner, the remaining essential quantities contained in (3.1) are calculated in turns as follows: $\pi(ij;\, ih,\, ih) = \frac{1}{4} p_i p_j p_h (1 + p_h)$ $(h \neq i, j)$, $\pi(ij;\, ih,\, ik) = \frac{1}{4} p_i p_j p_h p_k$ $(h,\, k \neq i, j;\; h \neq k)$, $\pi(ij;\, ih,\, jh) = \frac{1}{4} p_i p_j p_h (1 + p_h)$ $(h \neq i, j)$, $\pi(ij;\, ih,\, jk) = \frac{1}{4} p_i p_j p_h p_k$ $(h,\, k \neq i, j;\; h \neq k)$, $\pi(ij;\, hk, fg) = 0$ $(f,\, g \neq i, j)$.

Relations, similar to (3.15), stating $\pi(ij;\, ih,\, ih) = \pi(ij;\, ih,\, jh)$ $(h \neq i, j)$, $\pi(ij;\, ih,\, ik) = \pi(ij;\, ih,\, jk)$ $(h,\, k \neq i, j,\; h \neq k)$, may be noticed.

Summing up, the results will be put together into following table. In the table, the suffices i, j, h, k are supposed, as before, to be different each other, while f and g must differ only from i, j and may, in particular, coincide with h or k or each other.

Mother	First child	Second child			
		A_{ii}	A_{ih}	A_{ik}	A_{fg}
A_{ii}	A_{ii}	$\frac{1}{2}p_i^3(1+p_i)$	$\frac{1}{2}p_i^3p_h$	$\frac{1}{2}p_i^3p_k$	0
	A_{ih}	$\frac{1}{2}p_i^3p_h$	$\frac{1}{2}p_i^2p_h(1+p_h)$	$\frac{1}{2}p_i^2p_hp_k$	0
	A_{ik}	$\frac{1}{2}p_i^3p_k$	$\frac{1}{2}p_i^2p_hp_k$	$\frac{1}{2}p_i^2p_k(1+p_k)$	0
	A_{hk}	0	0	0	0
		p_i^3	$p_i^2p_h$	$p_i^2p_k$	0

Mother	First child	Second child					
		A_{ii}	A_{jj}	A_{ij}	A_{ih} | A_{jh}	A_{ik} | A_{jk}	A_{fg}
A_{ij}	A_{ii}	$\frac{1}{4}p_i^2p_j(1+p_i)$	$\frac{1}{4}p_i^2p_j^2$	$\frac{1}{4}p_i^2p_j(1+p_i+p_j)$	$\frac{1}{4}p_i^2p_jp_h$	$\frac{1}{4}p_i^2p_jp_k$	0
	A_{jj}	$\frac{1}{4}p_i^2p_j^2$	$\frac{1}{4}p_i^2p_j^2(1+p_j)$	$\frac{1}{4}p_ip_j^2(1+p_i+p_j)$	$\frac{1}{4}p_ip_j^2p_h$	$\frac{1}{4}p_ip_j^2p_k$	0
	A_{ij}	$\frac{1}{4}p_i^2p_j$ $\times(1+p_i+p_j)$	$\frac{1}{4}p_ip_j^2$ $\times(1+p_i+p_j)$	$\frac{1}{4}p_ip_j(p_i+p_j)$ $\times(1+p_i+p_j)$	$\frac{1}{4}p_ip_jp_h(p_i+p_j)$	$\frac{1}{4}p_ip_jp_k(p_i+p_j)$	0
	A_{ih} or A_{jh}	$\frac{1}{4}p_i^2p_jp_h$	$\frac{1}{4}p_ip_j^2p_h$	$\frac{1}{4}p_ip_jp_h(p_i+p_j)$	$\frac{1}{4}p_ip_jp_h(1+p_h)$	$\frac{1}{4}p_ip_jp_hp_k$	0
	A_{ik} or A_{jk}	$\frac{1}{4}p_i^2p_jp_k$	$\frac{1}{4}p_ip_j^2p_k$	$\frac{1}{4}p_ip_jp_k(p_i+p_j)$	$\frac{1}{4}p_ip_jp_hp_k$	$\frac{1}{4}p_ip_jp_k(1+p_k)$	0
	A_{hk}	0	0	0	0	0	0
		$p_i^2p_j$	$p_ip_j^2$	$p_ip_j(p_i+p_j)$	$p_ip_jp_h$	$p_ip_jp_k$	0

It is evidently seen that, corresponding to (1.16) of IV, the identities hold:

$$(3.21)\qquad \sum_{f\leq g}\pi(ij\,;\,hk,fg)=\pi(ij\,;\,hk) \text{ and } \sum_{h\leq k}\pi(ij\,;\,hk,fg)=\pi(ij\,;\,fg).$$

The results on phenotypes can be derived by a similar procedure as §1 of IV. If the gene $A_i\equiv A_{i_1}$ is dominant against and only against $A_{i_a}(2\leq a\leq\alpha)$, $A_h\equiv A_{h_1}$ and $A_f\equiv A_{f_1}$ against A_{h_b} $(2\leq b\leq\beta)$ and $A_{f_c}(2\leq c\leq\gamma)$ respectively, then the probability of mother-children combination $(A_i\,;\,A_h,\,A_f)$ on phenotypes is given by

$$(3.22)\qquad \Pi(i\,;\,h,\,f)=\sum_{a=1}^{\alpha}\sum_{b=1}^{\beta}\sum_{c=1}^{\gamma}\pi(i_1i_a\,;\,h_1h_b,\,f_1f_c).$$

If, in the combination, phenotypes composed of two different genes are interested, the circumstance becomes simpler; for instance, if mother is of such a heterozygote A_iA_j, we get, instead of (3.22), more briefly

$$(3.23)\qquad \Pi(ij\,;\,h,f)=\sum_{b=1}^{\beta}\sum_{c=1}^{\gamma}\pi(ij\,;\,h_1h_b,f_1f_c).$$

The results on mixed mother-child combinations, discussed in

§2 of IV, can also be extended to the two-children case. Making use of the similar notations as before, we obtain, corresponding to (2.12) of IV, the fundamental interrelations

$$(3.24)\quad \pi'(ij\,;\,hk,\ fg)/p_ip_j=[\pi(ij\,;\,hk,\ fg)/p_ip_j]^{(p_h,p_k,p_f,p_g)=(p_h',p_k',p_f',p_g')}\,.$$

Finally, corresponding to (2.19) of IV, the ratio defined by

$$(3.25)\qquad \frac{\pi(ij\,;\,hk,\ fg)}{\bar{A}_{fg}}=\begin{cases}\pi(ij\,;\,hk,ff)/p_f^2 & (g=f),\\ \pi(ij,\ hk,\ fg)/2p_fp_g & (g\neq f)\end{cases}$$

represents the probability a posteriori of the event that, for a fixed type A_{fg} of the second child, its mother with the first child of the A_{hk} is of the type A_{ij}. Similarly, the ratio defined by

$$(3.26)\quad \frac{\pi(ij\,;\,hk,\ fg)}{\bar{A}_{hk}\bar{A}_{fg}}=\begin{cases}\pi(ij\,;\,hh,ff)/p_h^2p_f^2 & (k=h,\ g=f),\\ \pi(ij\,;\,hh,fg)/2p_h^2p_fp_g & (k=h,\ g\neq f),\\ \pi(ij\,;\,hk,ff)/2p_hp_kp_f^2 & (k\neq h,\ g=f),\\ \pi(ij\,;\,hk,fg)/4p_hp_kp_fp_g & (k\neq h,\ g\neq f)\end{cases}$$

represents *the probability a posteriori of the event that, for the fixed types* A_{hk} *and* A_{fg} *of the first and the second children, their mother is of the type* A_{ij}.

—*To be continued*—