136. *On the Metrizable Condition.*[1]

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L. W. Cohen considered a neighborhood space S topologized by the neighborhood basis $\{U_\alpha(p)\}$ of each point p of S, where α is an element of a set A, such that:

I. $\bigcap_{\alpha\in A} U_\alpha(p) = \{p\}$,

II. For each $\alpha, \beta \in A$ and $p \in S$ there exists a $\gamma = \gamma(\alpha,\beta; p)\in A$ such that $U_\gamma(p)\subset U_\alpha(p)\frown U_\beta(p)$,

III. For each $\alpha\in A$ and $p\in S$ there exist $\lambda(\alpha)$, $\delta(p,\alpha)\in A$ such that, if $U_{\delta(p,\alpha)}(q)\frown U_{\lambda(\alpha)}(p) \neq 0$, then $U_{\delta(p,\alpha)}(q)\subset U_\alpha(p)$;

and he showed that such a space S can be imbedded in a complete space S^* [2]. He gave also a question as whether a Hausdorff space satisfying the first denumerability axiom and condition III is metrizable. This note is concerned with this question.

We consider the next condition IV instead of III,

IV. For each $\alpha\in A$ and $p\in S$ there exist $\lambda'(p, \alpha)$, $\delta'(p, \alpha)\in A$ such that, if $U_{\delta'(p,\alpha)}(q)\frown U_{\lambda'(p,\alpha)}(p) \neq 0$, then $U_{\delta'(p,\alpha)}(q) \subset U_\alpha(p)$.

Our result is

Theorem. *If the neighborhood space S satisfies the first denumerability axiom, i.e., the set of suffix of neighborhood basis can be taken to the set N of integers, and the above condition* I, II *and* IV, *then S is metrizable.*

As a Hausdorff space satisfies the condition I and II, and the condition III implies IV, this theorem gives the affirmative answer to the question.

To prove the theorem, we use

Frink's Theorem[3]. *A necessary and sufficient condition that a neighborhood space S be metrizable is that for every point* $p \in S$ *there exists a sequence of neighborhoods* $\{U_n(p)\}$, *monotone decreasing and whose logical product is* $\{p\}$, *selected from the original neighborhoods and equivalent to them, satisfying the condition*:

V. *For each* $n \in N$ *and* $p \in S$ *there exists* $m = m(p, n) \in N$ *such that* $m \geqq n$ *and if* $U_m(q)\frown U_m(p) \neq 0$ *then* $U_m(q)\subset U_n(p)$.

Let S satisfy the assumption of the theorem, i.e., for each point

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1) This was reported at the annual meeting of Math. Soc. of Japan in Oct. 1950, and has been completed by the encourag ment of Prof. T. Inagaki.

2) L. W. Cohen: On imbedding a space in a complete space, Duke Math. Jour. vol. 5(1935), p. 183.

3) A. H. Frink: Distance function and the metrization problem, Bull. Amer. Math. Soc. vol. 43 (1937), Theorem 4 p. 141.

p of S, there is a neighborhood basis $\{U_n(p)\}$ satisfying the conditions:

I*. $\bigcap_{n\in N} U_n(p) = \{p\}$,

II*. For each $m, n \in N$ and $p \in S$ there exists $l=l(m, n; p) \in N$ such that $U_l(p) \subset U_m(p) \frown U_n(p)$.

IV*. For each $n \in N$ and $p \in S$ there exist $h(p, n)$, $k(p, n) \in N$ such that, if $U_{h(p,n)}(q) \frown U_{k(p,n)}(p) \neq 0$, then $U_{h(p,n)}(q) \subset U_n(p)$.

We take $\{V_n(p)\}$ by induction as follows: $V_1(p) = U_1(p)$, and $V_n(p)$ is a neighborhood of p which is contained in $U_n(p) \frown V_{n-1}(p)$. Then, evidently, $\{U_n(p)\}$ and $\{V_n(p)\}$ are equivalent, and $\{V_n(p)\}$ is monotone decreasing and their logical product is $\{p\}$. For each $n \in N$ we select $n' = n'(p, n)$ such that $V_n(p) \supset U_{n'}(p)$ by equivalency of $\{U_n(p)\}$ and $\{V_n(p)\}$, and take $h(p, n')$ and $k(p, n')$ by IV* and set $m(p, n) = m_1(p, n', n) = Max(h(p, n'), k(p, n'), n)$. We shall prove that this $m(p, n)$ satisfies the condition V. Assume $V_{m(p,n)}(q) \frown V_{m(p,n)}(p) \neq 0$, then from $m(p, n) \geqslant k(p, n'), h(p, n')$, $V_{h(p,n')}(q) \frown V_{k(p,n')}(p) \neq 0$, and hence $U_{h(p,n')}(q) \frown U_{k(p,n')}(p) \neq 0$. As $h(p, n')$ and $k(p,n')$ $X$are defined by IV*, it implies $U_{h(p,n')}(q) \subset U_{n'}(p)$, and so $V_{m(p,n)}(q) \subset V_n(p)$. Thus $\{V_n(p)\}$ satisfies the condition V. Hence by Frink's Theorem we conclude that the space S is metrizable.

Here we notice that the condition II is necessary. To show this, it is sufficient to construct a space which satisfies the first denumerability axiom and I and IV and does not II, as a metrizable space satisfies the condition II.

Take X as the set of integers, and we set if p is even,

$$V_{2n}(p) = \{p-1, p\}, \quad V_{2n+1}(p) = \{p, p+1\}; \qquad (n = 1, 2, \dots),$$

if p is odd,

$$V_{2n}(p) = \{p, p, +1\}, \quad V_{2n+1}(p) = \{p-1, p\}; \qquad (n = 1, 2, \dots).$$

We take $\{V_n(p); n = 1, 2, \dots\}$ as a neighborhood basis at p, then X satisfies the first denumerability axiom and I and not II. In addition, X satisfies the condition IV*. Take $h(p, 2n) = 2 = k(p, 2n)$ and $h(p, 2n+1) = 1 = k(p, 2n+1)$ for every $p \in X$. If p is even and n is even, $V_{h(p,n)}(q) \frown V_{k(p,n)} \neq 0$ is same to $V_1(p) \frown V_1(q) \neq 0$, which implies $q = p+1$, and as q becomes odd, it follows that $V_{h(p,n)}(q) = V_1(q) = \{q-1, q\} = \{p, p+1\} = V_n(p)$. By the same manner, we can show that $h(p, n)$ and $k(p, n)$ satisfy the condition IV* for each $n \in N$ and $p \in X$.