135. On Linear Modulars.

By Sadayuki YAMAMURO.

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Let R be a modulared semi-ordered linear space[1] with a modular m. If R is semi-regular, we can introduce into R two sorts of norms, namely, *the first norm* $\|a\|$ $(a \varepsilon R)$ and *the second norm* $|||a|||$ $(a \varepsilon R)$, satisfying the condition

$$|||a||| \leqq \|a\| \leqq 2|||a||| \qquad (a \varepsilon R).$$

It is proved that, if m is linear or singular[2], then we have

$$(*) \qquad \|a\| = |||a||| \qquad (a \varepsilon R).$$

In this paper we will prove the converse, that is:

Theorem. If a modulared semi-ordered linear space R with a modular m is semi-regular and the condition (*) is always satisfied, then m is either linear or singular.

Suppose, in the sequel, that the condition (*) is satisfied and we denote the common value by $\|a\|$ $(a \varepsilon R)$.

Lemma 1. The first norm and the second norm by the conjugate modular $\overline{m}$ of m coincide.

Proof. The first norm by $\overline{m}$ is the conjugate norm of the second norm by m, and the second norm by $\overline{m}$ is the conjugate norm of the first norm by m. Hence our assertion is obtained.

Lemma 2. For a element a such that $\|a\| = 1+m(a)$, we have $m(a) = 0$.

Proof. Suppose $m(a) \geqq 1$. Then we have $m(a) \geqq \|a\|$ by the definition of the second norm, contradicting the assumption. Thus we have $m(a) < 1$, and hence $\|a\| \leqq 1$[3]. Therefore, from the assumption, we conclude $m(a) = 0$.

Lemma 3. If there is a simple domestic element a satisfying the condition $m(a) = 1$, then m is a linear modular on $[a]R$.

Proof. As a is simple and domestic, we can find a positive element $\bar{a}$ of the conjugate space $\overline{R}$ of R such that

$$\bar{a}(a) = \overline{m}(\bar{a}) + m(a), \quad \text{and} \quad [\bar{a}]^R = [a].$$

From this relation, we conclude $\|\bar{a}\| = \bar{a}(a) = \overline{m}(\bar{a})+1$, because, for the first norm $\|\bar{a}\|$ by $\overline{m}$, we have

$$\|\bar{a}\| = \sup_{m(x)\leqq 1} |\bar{a}(x)|, \quad \text{and} \quad |\bar{a}(x)| \leqq \overline{m}(\bar{a}) + m(x) \qquad (x \varepsilon R)$$

Thus we obtain $\overline{m}(\bar{a}) = 0$ by the previous lemma.

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1) H. Nakano: Modulared semi-ordered linear spaces. Tokyo Math. Book Series, Vol. I (1950), p. 153.
2) ibid., p. 184.
3) ibid., p. 181.

Next we prove that $\overline{m}$ is singular in $[\bar{a}]\overline{R}$. If this is not so, then we can find a number $\alpha > 1$ and a projector $[\bar{p}]$ such that

$$0 < \overline{m}(\alpha[\bar{p}]\bar{a}) \leqq 1 .$$

Putting $\bar{b} = (1-[\bar{p}])\,\bar{a} + \alpha\,[\bar{p}]\,\bar{a}$, we have

$$\overline{m}(\bar{b}) = \overline{m}((1-[\bar{p}])\,\bar{a}) + \overline{m}(\alpha\,[\bar{p}]\,\bar{a}) \leqq 1 ,$$

and hence

$$\| a \| = \sup_{\overline{m}(\bar{x}) \leqq 1} | \bar{x}(a) | \geqq \bar{b}(a) .$$

However this relation is impossible, because, by the condition $[\bar{a}]^R = [a]$, we have $[\bar{p}]\bar{a}(a) > 0$, and hence, as $\alpha > 1$,

$$\bar{b}(a) = (1-[\bar{p}])\bar{a}(a) + \alpha[\bar{p}]\bar{a}(a) > \bar{a}(a) = \| a \| .$$

Hence we have proved that $\overline{m}$ is singular in $[\bar{a}]\overline{R}$, that is, m is linear in $[a]R$.

Proof of the theorem. Let N be the totality of linear elements of R. Then m is linear in $[N]R$. For an element $x \varepsilon (1-[N])R$, $m(x) < +\infty$, we have $m(x) \leqq 1$. Because, if there is an element $x \varepsilon (1-[N])R$ such that $1 < m(x) < +\infty$, then we can find a number α and a projector $[p]$ such that $m(\alpha[p]x) = 1$ and $\alpha[p]x$ is simple and domestic, and consequently m is linear in $[[p]x]R$ as proved just above. This contradicts the definition of $(1-[N])R$. Hence, for any $\bar{x} \varepsilon \overline{R}(1-[N])$, we have

$$\begin{aligned} \overline{m}(\bar{x}) &= \sup_{x \varepsilon R} \{\bar{x}(x) - m(x)\} \\ &= \sup_{m(x) \leqq 1} \{\bar{x}(x) - m(x)\} \\ &\leqq \sup_{m(x) \leqq 1} \bar{x}(x) = \| \bar{x} \| , \end{aligned}$$

which shows that $\overline{m}$ is finite in $\overline{R}(1-[N])$. Then it is easily seen that $\overline{m}$ is linear in $\overline{R}(1-[N])$, that is, m is singular in $(1-[N])R$. If $[N]R \neq 0$, $(1-[N])R \neq 0$, then for $x \varepsilon [N]R$, $y \varepsilon (1-[N])R$ we see easily

$$\| x+y \| = \| x \| + \| y \| , \qquad ||| x+y ||| = \text{Max.} \{ ||| x |||, ||| y ||| \},$$

contradicting the assumption (*). Thus it is proved that

$$[N]R = 0 , \qquad \text{or} \qquad (1-[N])R = 0 .$$

Hence m is linear or singular on R.