7. On Some Representation Theorems in an Operator Algebra. III.

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(Comm. by K. KUNUGI, M.J.A., Jan. 12, 1952.)

When we discuss group algebras on a locally compact group, the notion of C*-algebra is very useful. But some group algebras are not always C*-algebra. Hence we shall introduce a notion of normed algebra including C*- and group algebras.

5. A normed*-algebra $\mathfrak{A}$ with the norm $\|\cdot\|$ over the complex number field is called with *D*-algebra* if $\mathfrak{A}$ has an approximate identity $\{e_\alpha\}$ (cf. Segal [1]) which is a directed set such as $\|e_\alpha\| \leq 1$ and $\|e_\alpha x - x\| \underset{\alpha}{\rightarrow} 0$ for any $x \in \mathfrak{A}$. Clearly C*- and L^1-group algebras are D*-algebras. $f(\cdot)$ is said to be *semi-trace* of $\mathfrak{A}$ if it is a linear functional (not always bounded) defined on a dense subalgebra generated by $\{xy;\ x, y \in \mathfrak{A}\}$ such that $f(xy)=f(yx)$, $f(x^*)=\overline{f(x)}$ $f(x^*x) \geq 0$ and $f((xy)^*xy) \leq \|x\|^2 f(y^*y)$. A semi-trace is *pure*, if it is not a linear combination of two linearly independent semi-traces of $\mathfrak{A}$. A representation $\{U_x, \mathfrak{H}\}$ is said to be *proper*, if the element $\xi \in \mathfrak{H}$ satisfying $U_x \xi = 0$ for all $x \in \mathfrak{A}$ is only the zero element O in $\mathfrak{H}$. Moreover we also define the properness for a two-sided representation $\{U_x, V_x, j, \mathfrak{H}\}$, if the one-sided representation $\{U_x, \mathfrak{H}\}$ or $\{V_x, \mathfrak{H}\}$ is proper. Denote the W*-algebras generated by $\{U_x;\ x \in \mathfrak{A}\}$ or $\{V_x;\ x \in \mathfrak{A}\}$ by $\boldsymbol{U}$ or $\boldsymbol{V}$ respectively.

Theorem 5. *Let τ be a semi-trace of a D*-algebra $\mathfrak{A}$. Then there corresponds a proper two-sided representation $\{U_x, V_x, j, \mathfrak{H}\}$ such that $\boldsymbol{U}=\boldsymbol{V}'$, $\boldsymbol{U}'=\boldsymbol{V}$ and $jAj=A^*$ for $A \in \boldsymbol{U} \frown \boldsymbol{V}$.*

Corollary 5.1. *If the D*-algebra $\mathfrak{A}$ is separable, then the semi-trace is a directed integral of pure semi-traces $\pi(\cdot, \lambda)$, $\lambda \bar{\in} N$ ($\sigma(\lambda)$-null set), with respect to a $\sigma(\lambda)$-measure.*

A two-sided representation $\{U_x, V_x, j, \mathfrak{H}\}$ is *strictly normal*, if there exists $\xi \in \mathfrak{H}$ such that $U_x\xi = V_x\xi$ for all $x \in \mathfrak{A}$ and $\{U_x\xi;\ x \in \mathfrak{A}\}$ span $\mathfrak{H}$. Then

Theorem 6. *If a D*-algebra $\mathfrak{A}$ has a strictly normal two-sided representation $\{U_x, V_x, j, \mathfrak{H}\}$, then the normalizing function is trace and conversly. This correspondence is one-to-one without equivalence. The generated W*-algebra $\boldsymbol{U}$ (or $\boldsymbol{V}$) has a complete trace and both in finite class (in the sense of J. Dixmier [3]).*

6. Motion in C*- or D*-algebra. The investigation of a motion in C*-algebra has been introduced by I. E. Segal (cf. [2]) which is one parameter group of automorphisms on $\mathfrak{A}$. We shall describe on a motion G of C*- or D*-algebra whose parameter is any group with a continuity, i.e., let $\mathfrak{A}$ be a D*-algebra and G be a group of automorphisms on $\mathfrak{A}$: for any $s \in G$ $(\alpha x+\beta y)^s=\alpha x^s+\beta y^s$, $x^{*s}=x^{s*}$ and $(xy)^s=x^s y^s$, and *G has a topology such as, for each $x \in \mathfrak{A}$ x^s is continuous on G.*

A trace or semi-trace τ of $\mathfrak{A}$ is said to be *stationary,* if $\tau(x^s y^s)=\tau(xy)$ for all $x, y \in \mathfrak{A}$ and $s \in G$. A stationary trace or semi-trace is *ergodic,* if it is not a positive linear combination of linearly independent such two other traces or semi-traces respectively.

Theorem 7. *Any stationary trace $\tau(\cdot)$ is represented by a directed integral of ergodic traces $\pi(\cdot, \lambda)$, $\lambda \overline{\in} N$, with respect to a suitable weight function $\sigma(\lambda)$:*

$$\tau(x)=\int_R \pi(x, \lambda)\, d\sigma(\lambda) \qquad \textit{for all } x \in \mathfrak{A}.$$

Finaly we shall state that the decomposition of invariant regular measure on a locally compact space with a group of homeomorphisms. Let Ω be a locally compact Hausdorff space and G be a group of homeomorphisms on Ω. C_∞ or C_0 denote the class of all continuous functions on Ω vanishing at infinity or zero out-side of compact sets of Ω respectively. For all $x \in C_\infty$ or C_0, if we put $\|x\|=\sup_{\omega\in\Omega}|x(\omega)|$, then C_∞ or C_0 is C*- or D*-algebra. A homeomorphism h on Ω reduces an automorphism on C_∞ or C_0 by $x^h(\omega)=x(h\omega)$ for $x \in C_\infty$ or C_0 and $\omega \in \Omega$, and conversely. Hence G can be considered as a group of automorphisms on C_∞ or C_0. We can introduce a topology into G for which G is motion of C_∞ or C_0.

Let $\mu(\cdot)$ be a regular measure on Ω invariant under G. Putting

$$\tau(x)=\int_\Omega x(\omega)\, d\mu(\omega) \qquad \text{for } x \in C_0:$$

$\tau(\cdot)$ is a stationary semi-trace of C_0. Applying the Corollary 5.1, we have

Theorem 8. *Let Ω be a locally compact Hausdorff space satisfying the second countable axiom with a group G of homeomorphisms on Ω, and $\mu(\cdot)$ be a regular measure of Ω invariant under G. Then there exists a system of ergodic measures $\mu(\cdot, \lambda)$ of Ω, $\lambda \overline{\in} N(\sigma(\lambda)$-null set), such that for all $x \in C_0$*

$$\text{(1)} \qquad \int_\Omega x(\omega)\, d\mu(\omega)=\int_R \int_\Omega x(\omega)\, d\mu(\lambda, \omega)\, d\sigma(\lambda)$$

and for all compact sets K in Ω

$$\text{(2)} \qquad \mu(K)=\int_R \mu(K, \lambda)\, d\sigma(\lambda)$$

where $\sigma(\lambda)$ *is a suitable weight function.*

The proofs of all theorems in this paper will apper in elsewhere.

References.

[1] I. E. Segal: Irreducible representations of operator algebra. Bull. Amer. Math. Soc., 48 (1947), 73-88.

[2] I. E. Segal: A class of operator algebras which are determined by group. Duke Math. J., 18 (1951), 221-265.

[3] J. Dixmier: Les Anneaux d'Operateus de class Finie. Ann. Sci. de L'Ecole Norm. Sup. Paris (1949), 209-261.

Appendix for the definition of semi-trace $\tau(x)$ in §5: there exists a sequence $\{e_{\alpha n}\}$ $(\subset\{e_\alpha\})$ dependently on $x\in\mathfrak{A}$ such that $\tau(e_{\alpha n}^* x^* x e_{\alpha n})\rightarrow\tau(x^*x)$ as $n\rightarrow\infty$.