6. *Cousin Problems for Ideals and the Domain of Regularity. II.*

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1. In the previous paper[1], we have considered the Cousin problems for ideals. The second Cousin problem *for ideals* is always solvable in a domain of regularity, but the "classical" or "functional" second Cousin problem remains still unperfectly solved. Mr. K. Stein[2] has set up a necessary and sufficient condition for the solvability of the latter, but, I think, his condition is the one not for the domain, but for the given Cousin distribution.

In the present note, we shall prove a necessary and sufficient condition for the solvability of the functional second Cousin problem in a domain of regularity. Although our condition seems to be over complicated when compared to the complicacy of the original Cousin problem, I believe it will be applicable to the theory of ideals or varieties in a domain of regularity in which the solvability of the second Cousin problem has already been established.

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2. First we will arrange the notions used later. The terminologies not defined here are all found in my previous note.[1]

We always consider the domains in the space of n complex variables $z_1, \dots, z_n$ which we denote by z only. When we use the word "domain of regularity", it is always supposed to be univalent and finite.

Definition 1. An ideal $\mathfrak{J}$ in a domain G is said to be *locally simple* if the punctual ideal $\mathfrak{J}_a$ generated by $\mathfrak{J}$ at any point a of G is always principal.

Definition 2. Two ideals $\mathfrak{J}$ and $\mathfrak{P}$ in a domain G are said to be *locally equivalent* if they generate quite the same punctual ideals at every point of G.

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1) S. Hitotumatu: Cousin problems for ideals and the domain of regularity. Kōdai Math. Sem. Reports, vol. **3**, Nos. 1/2 (1951), 26-32.

2) K. Stein: Topologische Bedingungen für die Existenz analytischer Funktionen komplexer Veränderlichen zu vorgegebene Nullstellenflächen. Math. Ann. **117** (1941), 727-757. See also, Math. Ann. **123** (1951), 202-222.

Definition 3.[3] Let $\mathfrak{J}$ be an ideal in an open set G. We denote by $\overline{\mathfrak{J}}$ the ideal consisting of the functions regular in G which are, in every compact subset of G, uniform limits of functions of $\mathfrak{J}$. This ideal $\overline{\mathfrak{J}}$ is called the *closure* of $\mathfrak{J}$. $\mathfrak{J}$ is said to be *closed* when $\mathfrak{J}=\overline{\mathfrak{J}}$.

It is evident that $\overline{\mathfrak{J}}$ is closed.

Definition 4. Two functions $\varphi(z)$ and $\psi(z)$ regular in a domain G are said to be *equivalent with respect to division*, or simply D *equivalent* in G, if the quotient φ/ψ is regular and non-zero in G.

Definition 5. Suppose that to every point a of a set E, there correspond a neighborhood $U(a)$ and a function $\varphi_a(z)$ regular in $U(a)$, such that φ_a and φ_b are D-equivalent in $U(a)\frown U(b)$, unless this intersection is empty. Such a system $\{U(a);\ \varphi_a\}$ is called a *functional second Cousin distribution* or f.s.C.d. in E.

We also denote the "second Cousin distribution for ideals"[4] by i.s.C.d. for the simplicity's sake.

Definition 6. Suppose that to a given f.s.C.d. $\{U(a);\ \varphi_a\}$ in E, there exists a function $\Phi(z)$ regular in E, which is D-equivalent to φ_a in $U(a)$ for every $a \in E$. Such a function Φ is called the *solution* of the given f.s.C.d.

3. The lemmas used later are given in the following. We omit the proofs, but they are found in the cited places.

Lemma 1.[5] To every i.s.C.d. in a domain of regularity, there always exists a *closed* ideal[6], which is the solution of the given i.s.C.d.

Lemma 2.[7] If two *closed* ideals in a domain of regularity are locally equivalent, they must coincide with each other.

Lemma 3.[8] Let $\mathfrak{J}_a$ be a punctual ideal at a, and a function f be regular in a neighborhood $U(a)$. If there exist functions $f^{(\nu)} \in \mathfrak{J}_a$ such that they are all regular in $U(a)$ and converge uniformly to f in $U(a)$, then f itself belongs to $\mathfrak{J}_a$.

Lemma 4.[9] If two ideals $\mathfrak{J}$ and $\mathfrak{P}$ in a neighborhood $V(a)$ generate the same punctual ideals at a, then there exists a neigh-

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3) Due to H. Cartan: Idéaux et modules de fonctions analytiques de variables complexes. Bull. Soc. Math. France **78** (1950), 29-64: § 30, p. 60.

4) Loc. cit. Hitotumatu 1), p. 27. In other words, the system of punctual ideals generated by a "faisceau cohérent" in H. Cartan's sense.

5) Loc. cit. H. Cartan 3), Theorem 3 ter, § 29, p. 60.

6) The solution-ideal is unique: this is a direct consequence of Lemma 2.

7) Loc. cit. H. Cartan 3), Theorem 4 ter and its Corollary, § 30, p. 60.

8) H. Cartan: Idéaux de fonctions analytiques de n variables complexes. Ann. École. Norm. Sup. **61** (1944), 149-197: First Corollary of Theorem α, p. 194.

9) Loc. cit. H. Cartan 8), Second Corollary of Theorem α, p. 194.

borhood $U(a) \subset V(a)$, in which $\mathfrak{J}$ and $\mathfrak{P}$ are locally equivalent.

Lemma 5.[10] An ideal in a domain of regularity with finite bases is closed.

4. Our Theorem is the following:

Theorem 1. *The condition for the existence of the solution for any given f.s.C.d. in a domain of regularity, is that every closed, locally simple ideal in it is principal.*

Proof. Necessity: We take a closed, locally simple ideal $\mathfrak{J}$ in the domain G. By our assumption, the punctual ideal $\mathfrak{J}_a$ generated by $\mathfrak{J}$ at a has a base consisting of a simple element, say φ_a, which is regular in a neighborhood $V(a)$. Since the ideal $\mathfrak{J}$ and $\{\varphi_a\} = \mathfrak{P}^{(a)}$ in $V(a)$ generate the same punctual ideals at a, there exists a neighborhood $U(a) \subset V(a)$ satisfying the condition of Lemma 4. We consider the system of these neighborhoods $U(a)$ and the functions φ_a. If $U(a) \frown U(b)$ is not empty, $\{\varphi_a\} = \mathfrak{P}^{(a)}$ and $\{\varphi_b\} = \mathfrak{P}^{(b)}$ generate the same ideals $\mathfrak{J}_c$ at every point c of $U(a) \frown U(b)$, and so φ_a and φ_b are D-equivalent in $U(a) \frown U(b)$. Therefore we have a f.s.C.d. $\{U(a); \varphi_a\}$ in G, and then we have its solution Φ by hypothesis. Since the ideal $\mathfrak{P} = \{\Phi\}$ in G has finite bases, it is closed by Lemma 5. On the other hand, $\mathfrak{P}$ generates the punctual ideals $\mathfrak{P}_a = \mathfrak{P}_a^{(a)} = \mathfrak{J}_a$ at $a \in G$. Therefore $\mathfrak{P}$ and $\mathfrak{J}$ are locally equivalent, and so by Lemma 2, they must coincide with each other. Hence $\mathfrak{J}$ is principal.

Sufficiency: For a given f.s.C.d. $\{U(a); \varphi_a\}$ in G, we construct the punctual ideal $\mathfrak{J}_a$ generated by φ_a at a. It is evident that the system $\{U(a); \mathfrak{J}_a\}$ is an i.s.C.d. in G, by definitions. Then we have its solution-ideal $\mathfrak{J}$ by Lemma 1, and we may assume that $\mathfrak{J}$ is closed. Also $\mathfrak{J}$ is locally simple, because it generates the principal ideal $\mathfrak{J}_a = \{\varphi_a\}$ at a by our assumption. Therefore $\mathfrak{J}$ is principal by hypothesis. Let Φ be its base. At every point $c \in U(a)$, $\mathfrak{J}$ generates the punctual ideal $\mathfrak{J}_c = \{\varphi_c\}$, and so Φ/φ_c is regular and non-zero at c. On the other hand, φ_c/φ_a is regular and non-zero at c by our assumption, hence Φ and φ_a are D-equivalent in $U(a)$. This means that Φ is the solution of the given f.s.C.d. Thus our Theorem is proved.

Corollary 1. *Suppose that there always exists the solution for any given f.s.C.d. in a domain of regularity G.* (For example, let G be the whole finite space, or a cylindrical domain whose components are simply-connected except at most one.) *Then every analytic variety in G with local complex dimension $n-1$ at any*

10) Loc. cit. H. Cartan 3), Theorem 11, § 32, p. 62.

point on it, can be represented as the zero-manifold of a function, regular in G.

Corollary 2. *If $n=1$, every closed ideal in a domain is principal.*

Because, if $n=1$, every punctual ideal is principal.

5. In this section we shall give a property of the closure of the ideal, which is a generalization of Lemma 2. We express this only for ideals, but it is also valid for higher-dimensional modules.[11]

Theorem 2. *The closures of two ideals $\mathfrak{J}$ and $\mathfrak{P}$ in a domain of regularity coincide with each other if and only if $\mathfrak{J}$ and $\mathfrak{P}$ are locally equivalent.*

First we put the following:

Lemma 6. *In a domain of regularity, an ideal $\mathfrak{J}$ and its closure $\overline{\mathfrak{J}}$ are locally equivalent.*

If this had been proved, the necessity of Theorem 2 would be evident. Conversely, if $\mathfrak{J}$ and $\mathfrak{P}$ are locally equivalent, their closures $\overline{\mathfrak{J}}$ and $\overline{\mathfrak{P}}$ must be also locally equivalent by this Lemma, and so they coincide with each other by Lemma 2.

Proof of Lemma 6: Since it is evident that $\mathfrak{J}_a \subset \overline{\mathfrak{J}}_a$, we have only to prove the converse. If $\varphi \in \overline{\mathfrak{J}}$, we have

$$\varphi = \sum_{j=1}^{s} a_j f_j$$

where $f_j \in \overline{\mathfrak{J}}$ and a_j are regular at a. There exists a neighborhood $U(a)$ completely interior to G, in which all the functions a_j ($j=1, \dots, s$) are regular and bounded, and we have functions $g_j^{(\nu)} \in \mathfrak{J}$, such that $\{g_j^{(\nu)}\}_{\nu=1}^{\infty}$ converge uniformly to f_j in every compact subset of G. From these assumptions

$$\sum_{j=1}^{s} a_j g_j^{(\nu)} \in \mathfrak{J}_a$$

are regular in $U(a)$, and converge uniformly to

$$\sum_{j=1}^{s} a_j f_j = \varphi$$

in $U(a)$. Hence we have $\varphi \in \mathfrak{J}_a$ by Lemma 3. Thus our statement is proved.

11) This is a notion introduced by H. Cartan: loc. cit. 8) or 3).