1. A Note on Symmetric Algebras.

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The main purpose of the present note is to prove the following theorem[1] by a new method.

Theorem 1. *An algebra A over an algebraically closed field is symmetric if and only if its basic algebra is symmetric.*

As an application, we can show that absolutely uni-serial algebras are symmetric.

In what follows we assume always that A is an algebra with unit element over an algebraically closed field K. Let $S(a)$ and $R(a)$ be the left and the right regular representations of A, formed by means of a basis (u_i). A is called a Frobenius algebra if $S(a)$ and $R(a)$ are similar:

$$(1) \qquad S(a) = P^{-1}R(a)P.$$

In particular, A is called a symmetric algebra when the matrix P can be chosen as a symmetric matrix[2].

Let $A = A^* + N$ be a splitting of an algebra A into a direct sum of a semisimple subalgebra A^* and the radical N of A. We shall denote by

$$A^* = A_1^* + A_2^* + \cdots\cdots + A_n^*$$

the unique splitting of A^* into a direct sum of simple invariant subalgebras. Let $e_{\kappa,\alpha\beta}$ $(\alpha, \beta = 1, 2, \ldots, f(\kappa))$ be a set of matrix units for the simple algebra A_κ^*. We set $e = \sum e_{\kappa,11}$. Then eAe is an algebra with unit element e, which is called the *basic algebra*[3] of A. As one can easily see, the radical of eAe is $eAe \frown N = eNe$ and eAe/eNe is direct sum of fields.

Let now

$$(2) \qquad A = A_1 \supset A_2 \supset \cdots\cdots \supset A_t \supset (0)$$

be a composition series for A considered as an (A, A) space. Then corresponding to (2), we obtain a composition series for eAe considered as an (eAe, eAe) space

$$(3) \qquad eAe = eA_1e \supset eA_2e \supset \cdots\cdots \supset eA_te \supset (0)$$

1) See Nesbitt and Scott [5] p. 549.

2) Nesbitt and Nakayama [4].

3) Nesbitt and Scott [5].

Since K is algebraically closed, the rank of eAe is t. Let the composition factor group A_u/A_{u+1} be of type (κ_u, λ_u), $(u=1,2,\ldots\ldots,t)$. Then we can choose a basis $b_1, b_2, \ldots\ldots, b_t$ of eAe corresponding to (3) such that $b_u \in eA_ue$, $b_u \notin eA_{u+1}e$ and $e_{\kappa_u, 11} b_u e_{\lambda_u, 11} = b_u$. Then the elements

$$e_{\kappa_u, \alpha 1} b_u e_{\lambda_u, 1\beta} \tag{4}$$

$(u=1,2,\ldots,t;\ \alpha=1,2,\ldots,f(\kappa_u);\ \beta=1,2,\ldots,f(\lambda_u))$ form a K-basis of A. This basis is called the *Cartan basis*[4] of A. Let us denote by c_{uvw} the multiplication constants of the elements b_u $(u=1,2,\ldots,t)$,

$$b_u b_v = \sum c_{uvw} b_w\,. \tag{5}$$

If $\lambda_u \neq \kappa_v$, then $c_{uvw}=0$ for every w. Further if $\lambda_u = \kappa_v$, then $c_{uvw}=0$ for either $\kappa_u \neq \kappa_w$ or $\lambda_v \neq \lambda_w$. Let $S_0(eae)$ and $R_0(eae)$ be the left and the right regular representations of eAe, formed by means of (b_u). Then

$$S_0(b_u) = (c_{uvw})_{wv}\,, \quad R_0(b_v) = (c_{uvw})_{uw}\,. \tag{6}$$

Let us assume that A is a symmetric algebra and let $S(a)$ and $R(a)$ be the regular representations of A, formed by means of the Cartan basis $(e_{\kappa_u, \alpha 1} b_u e_{\lambda_u, 1\beta})$;

$$\begin{cases} a(e_{\kappa_u, \alpha 1} b_u e_{\lambda_u, 1\beta}) = (e_{\kappa_u, \alpha 1} b_u e_{\lambda_u, 1\beta}) S(a) \\ (e_{\kappa_u, \alpha 1} b_u e_{\lambda_u, 1\beta}) a = (e_{\kappa_u, \alpha 1} b_u e_{\lambda_u, 1\beta}) R'(a) \end{cases} \tag{7}$$

where $R'(a)$ denotes the transpose of matrix $R(a)$. There exists a symmetric non-singular matrix P such that $S(a) = P^{-1}R(a)P$. If we set

$$(h_{u, \beta\alpha}) = (e_{\kappa_u, \alpha 1} b_u e_{\lambda_u, 1\beta}) P^{-1}, \tag{8}$$

then $(e_{\kappa_u, \alpha 1} b_u e_{\lambda_u, 1\beta})$ and $(h_{u, \beta\alpha})$ are the *quasi-complementary bases*[5] of A. Hence we have[6]

$$h_{u, \beta\alpha} = e_{\lambda_u, \beta 1} d_u e_{\kappa_u, 1\alpha} \tag{9}$$

where (d_u), $(u=1,2,\ldots,t)$ is a basis of eAe and d_u is of type (λ_u, κ_u), that is,

$$d_u = e_{\lambda_u, 11} d_u e_{\kappa_u, 11}\,. \tag{10}$$

Since $(e_{\kappa_u, \alpha 1} b_u e_{\lambda_u, 1\beta})$ and $(e_{\lambda_u, \beta 1} d_u e_{\kappa_u, 1\alpha})$ are quasi-complementary, we have

4) Nesbitt [3], Scott [7]. Cf. also Nesbitt and Scott [5].
5) See Brauer [1].
6) Osima [6].

$$(11)\qquad \begin{cases} a\,(e_{\lambda_u,\beta 1}\, d_u\, e_{\kappa_u, 1\alpha}) = (e_{\lambda_u,\beta 1}\, d_u\, e_{\kappa_u, 1\alpha})\, R\,(a) \\ (e_{\lambda_u,\beta 1}\, d_u\, e_{\kappa_u, 1\alpha})\, a = (e_{\lambda_u,\beta 1}\, d_u\, e_{\kappa_u, 1\alpha})\, S'\,(a)\,. \end{cases}$$

Proof of Theorem 1. Let A be symmetric and let $(e_{\kappa_u,\alpha 1}\, b_u\, e_{\lambda_u, 1\beta})$ be the Cartan basis of A. We arrange the elements of the basis $(e_{\kappa_u,\alpha 1}\, b_u\, e_{\lambda_u, 1\beta})$ as follows:

$$(12)\qquad b_1,\ b_2,\ \ldots\ldots,\ b_t,\ \ldots\ldots,\ e_{\kappa_u,\alpha 1}\, b_u\, e_{\lambda_u, 1\beta},\ \ldots\ldots$$

Then from (9) and (10) it follows that the elements of the basis $(e_{\lambda_u,\beta 1}\, d_u\, e_{\kappa_u, 1\alpha})$ are ordered as follows:

$$(13)\qquad d_1,\ d_2,\ \ldots\ldots,\ d_t,\ \ldots\ldots,\ e_{\lambda_u,\beta 1}\, d_u\, e_{\kappa_u, 1\alpha},\ \ldots\ldots$$

Hence, by (8), (9) and (13) we have

$$P^{-1} = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix},$$

where P_1 is a matrix of degree t. Since P^{-1} is symmetric, P_1 is also a symmetric, non-singular matrix. Further, as one can easily see, for any eae in eAe

$$S\,(eae) = \begin{pmatrix} S_0\,(eae) & 0 \\ 0 & 0 \end{pmatrix},\quad R\,(eae) = \begin{pmatrix} R_0\,(eae) & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $S_0\,(eae) = P_1 R_0\,(eae)\, P_1^{-1}$ and eAe is symmetric.

Now we prove the converse. Suppose that eAe is symmetric: $S_0(eae) = Q^{-1} R_0\,(eae)\, Q$ where Q is a symmetric matrix. We set $d_u = (b_u)\, Q^{-1}$. Then (b_u) and (d_u) are quasi-complementary bases of eAe. Hence

$$(14)\qquad \begin{cases} eae\,(d_u) = (d_u)\, R_0\,(eae) \\ (d_u)\, eae = (d_u)\, S_0'\,(eae)\,. \end{cases}$$

Then, by (6)

$$(15)\qquad \begin{cases} d_w b_u = \sum_v c_{uvw} d_v \\ b_v d_w = \sum_u c_{uvw}\, d_u\,. \end{cases}$$

The elements $e_{\lambda_u,\beta 1}\, d_u\, e_{\kappa_u, 1\alpha}$ ($u = 1, 2, \ldots\ldots, t$; $\alpha = 1, 2, \ldots\ldots, f(\kappa_u)$; $\beta = 1, 2, \ldots\ldots, f(\lambda_u)$) form the Cartan basis of A. (5) yields

$$\begin{aligned} e_{\kappa_u,\alpha 1}\, b_u\, e_{\lambda_u, 1\beta}\cdot e_{\kappa_v,\mu 1}\, b_v\, e_{\lambda_v, 1\nu} &= \sum c_{uvw}\, e_{\kappa_u,\alpha 1}\, b_w\, e_{\lambda_v, 1\nu} && \lambda_u = \kappa_v\,,\ \beta = \mu \\ &= 0 && \text{in other cases.} \end{aligned}$$

Here, if $c_{uvw} \neq 0$, then $\kappa_w = \kappa_u$ and $\lambda_w = \lambda_v$. Hence, from (15) we obtain

$$(16)\qquad \begin{cases} e_{\lambda_v,\nu 1}\, d_w\, e_{\kappa_u, 1\alpha}\cdot e_{\kappa_u,\alpha 1}\, b_u\, e_{\lambda_u, 1\beta} = \sum_v c_{uvw}\, e_{\lambda_v,\nu 1}\, d_v\, e_{\lambda_u, 1\beta} \\ e_{\kappa_v,\mu 1}\, b_v\, e_{\lambda_v, 1\nu}\cdot e_{\lambda_v,\nu 1}\, d_w\, e_{\kappa_u, 1\alpha} = \sum_u c_{uvw}\, e_{\kappa_v,\mu 1}\, d_u\, e_{\kappa_u, 1\alpha}\,. \end{cases}$$

This implies that

$$(17)\qquad \begin{cases} a(e_{\lambda_u,\beta 1}\, d_u\, e_{\kappa_u, 1\alpha}) = (e_{\lambda_u,\beta 1}\, d_u\, e_{\kappa_u, 1\alpha})\, R(a) \\ (e_{\lambda_u,\beta 1}\, d_u\, e_{\kappa_u, 1\alpha})a = (e_{\lambda_u,\beta 1}\, d_u\, e_{\kappa_u, 1\alpha})\, S'(a), \end{cases}$$

whence A is symmetric.

Theorem 2. *Let A be a uni-serial algebra over an algebraically closed field. Then A is symmetric.*

Proof. We may assume without loss of generality that A is primary: $A = A^* + N$ where A^* is a simple subalgebra. We denote by $e_{\alpha\beta}$ $(\alpha, \beta = 1, 2, \ldots\ldots, f)$ a set of matrix units for the simple algebra A^*. The radical N is a principal ideal: $N = dA = Ad$ where $e_{\alpha\alpha} d = de_{\alpha\alpha}$, $(\alpha = 1, 2, \ldots\ldots, f)$. The basic algebra eAe, $(e = e_{11})$, is also uni-serial and $eNe = eAede = edeAe$ where $ed = de = ede$. Let ρ be the exponent of N, that is, $N^{\rho-1} \neq 0$, $N^\rho = 0$. Then the elements

$$(18)\qquad e,\ de,\ d^2e, \ldots\ldots,\ d^{\rho-1}e$$

form a basis $(d^\kappa e)$ of eAe. This impries that eAe is commutative. If we set

$$(19)\qquad (ed^{\rho-1},\ ed^{\rho-2}, \ldots,\ ed,\ e) = (e,\ de, \ldots,\ d^{\rho-2}e,\ d^{\rho-1}e)\,P,$$

then P is a symmetric, non-singular matrix. Hence (ed^λ) is a basis of eAe. Let $S_0(eae)$ and $R_0(eae)$ be the left and the right regular representations of eAe, formed by means of the basis $(d^\kappa e)$. Then $R_0(eae)$ is the left regular representation of eAe, formed by means of the basis (ed^λ) whence $S_0(eae) = P^{-1}R_0(eae)P$. This implies that eAe is symmetric. Then it follows from Theorem 1 that A is symmetric.

Corollary. *Absolutely uni-serial algebras are symmetric.*

References.

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