51. *Probability-theoretic Investigations on Inheritance. IX₁. Non-Paterity Concerning Mother-Children Combinations.*

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1. Several problems to be discussed.

The problems discussed in two preceding chapters[1] have concerned exclusively the probabilities of proving non-paternity in case where a pair of a mother and her one child is presented. According to the nature of problems, the use has been made of the probability of mother-child combination obtained in §§ 1 and 2 of IV. Now, with respect to mother-children combination, there arise analogous problems which will be discussed in the present chapter. For the sake of brevity we restrict ourselves to two-children case; generalization to several children case will be left to the reader.

Various sorts of problems are now to be considered; for instance, given two children possessing a common mother, to determine a probability of proving non-paternity against a distinguished child at any rate, against a distinguished child alone, against at least one child, or against both children. Moreover, it may also be considered, for instance, to determine a probability in which the father of first child can assert the non-paternity against second child. Two cases are distinguished according to both children possessing father also in common or not; namely, to cases discussed in § 3 of IV or § 5 of IV. The last-mentioned problem concerns, of course, the latter, while the remaining problems mentioned above may concern either of cases.

In the present chapter we shall discuss the problems concerning two children belonging to the same family, i.e., possessing a father also in common. Hence, the use will be made of the results obtained

1) Y. Komatu, Probability-theoretic investigations on inheritance. I. Distribution of genes; II. Cross-breeding phenomena; III. Further discussions on cross-breeding; IV. Mother-child combinations; V. Brethren combinations; VI. Rate of danger in random blood transfusion; VII. Non-Paternity problems; VIII. Further discussions on non-paternity problems. Proc. Jap. Acad. **27** (1951), I. 371-377; II. 378-383, 384-387; III. 459-465, 466-471, 472-477, 478-483; IV. 587-592, 593-597, 598-603, 605-610, 611-614, 615-620; V.; **28** (1952), VI. 54-58; VII. 102-104, 105-108, 109-111, 112-115, 116-120, 121-125; VIII. 161-164, 165-168, 169-171. These papers will be referred to as I; II; III; IV; V; VI; VII; VIII.

in § 3 of IV. Problems concerning to two children possessing a mother alone in common will be postponed into a subsequent chapter.

2. Non-paternity against a distinguished child at any rate.

Let a set consisting of a mother and her two children be given; the order of children being taken into account. As in § 3 of IV, we call them the first and the second child according to their order. Then, the problem is to determine the probability in which a man, chosen at random with respect to type of the inherited character in question, can assert his non-paternity. Since it is indifferent from our present view-point whether a distinguished child is the first or the second, we may and shall assume that it is the second child, unless the contrary is stated.

Now, let a given set consist of a mother A_{ij} and first child A_{hk} and second A_{fg}. The probability of proving non-paternity of a man chosen at random against the second child will then be denoted by

$$(2.1) \qquad V(ij;\, fg),$$

in conformity with the notation introduced in (2.1) of VII; the quantity being independent of the type of first child. Here also, only the cases are significant where there exist common suffices between i, j and h, k and between i, j and f, g.

On the other hand, the probability of the triple is given by

$$(2.2) \qquad \pi(ij;\, hk,\, fg),$$

the quantity introduced in (3.3) of IV. Hence, *the probability of proving non-paternity against second child indifferent to first child* is then, similarly to (2.3) of VII, given by

$$(2.3) \qquad P(ij;\, hk,\, fg)=\pi(ij;\, hk,\, fg)\, V(ij;\, fg).$$

If we sum up the last quantities over all the possible indices h, k, then we get, in view of the second relation (3.21) of IV, a relation

$$(2.4) \qquad \sum_{h\leqq k} P(ij;\, hk,\, fg)=\pi(ij;\, fg)\, V(ij;\, fg),$$

of which the left-hand side represents nothing but the quantity $P(ij; fg)$ introduced in (2.3) of VII. This is an evident fact. Indeed, the elimination of the type of first child, which is indifferent, must lead, by means of the summation process, to a previous problem concerning pair of a mother and her second child alone. But, summing up the quantities (2.3) over all the possible indices f, g, then we get the partial sum of probabilities of proving non-paternity, a type of first child being indifferent, against second child with mother A_{ij} accompanying first child A_{hk}, which will be denoted by

$$(2.5) \qquad I(ij;\, hk)=\sum_{f\leqq g} P(ij;\, hk,\, fg).$$

Values of $\pi(ij; hk, fg)$ and of $V(ij; fg)$ having already been determined in § 3 of IV and in § 2 of VII respectively, we now can calculate the value of $I(ij; hk)$ thus defined. We first consider the case where a mother and her first child are both of the same homozygote. We then obtain

$$I(ii; ii) = P(ii; ii, ii) + \sum_{k \neq i} P(ii; ii, ik) = \tfrac{1}{2} p_i^3 (1+p_i)(1-p_i)^2 + \sum_{k \neq i} \tfrac{1}{2} p_i^3 p_k (1-p_k)^2 = \tfrac{1}{2} p_i^3 (2 - 2S_2 + S_3 - 2p_i + p_i^2). \tag{2.6}$$

The remaining cases can also be treated in a similar manner, yielding the following results:

$$I(ii; ih) = P(ii; ih, ii) + P(ii; ih, ih) + \sum_{k \neq i, h} P(ii; ih, ik) = \tfrac{1}{2} p_i^2 p_h (2 - 2S_2 + S_3 - 2p_h + p_h^2) \quad (h \neq i); \tag{2.7}$$

$$\begin{aligned} I(ij; ii) &= P(ij; ii, ii) + P(ij; ii, jj) + P(ij; ii, ij) + \sum_{k \neq i, j} (P(ij; ii, ik) + P(ij; ii, jk)) \\ &= \tfrac{1}{4} p_i^2 p_j (2(2 - 2S_2 + S_3) - 2(2p_i + p_j) + 2p_i^2 + p_j^2 - 2p_i p_j + 3p_i p_j (p_i + p_j)) \quad (i \neq j), \end{aligned} \tag{2.8}$$

$$\begin{aligned} I(ij; ij) &= P(ij; ij, ii) + P(ij; ij, jj) + P(ij; ij, ij) + \sum_{k \neq i, j} (P(ij; ij, ik) + P(ij; ij, jk)) \\ &= \tfrac{1}{4} p_i p_j (2(2 - 2S_2 + S_3)(p_i + p_j) - 4(p_i^2 + p_j^2) - 4p_i p_j + 2(p_i^3 + p_j^3) - 2p_i p_j (p_i + p_j) + 3p_i p_j (p_i + p_j)^2) \quad (i \neq j), \end{aligned} \tag{2.9}$$

$$\begin{aligned} I(ij; ih) &= P(ij; ih, ii) + P(ij; ih, jj) + P(ij; ih, ij) + P(ij; ih, ih) + P(ij; ih, jh) + \sum_{k \neq i, j, h} (P(ij; ih, ik) + P(ij; ih, jk)) \\ &= \tfrac{1}{4} p_i p_j p_h (2(2 - 2S_2 + S_3) - 4p_i p_j + 3p_i p_j (p_i + p_j) - 4p_h + 4p_h^2) \quad (i \neq j;\ h \neq i, j). \end{aligned} \tag{2.10}$$

Thus, the possible cases have essentially been worked out. In particular, the quantities $I(ii; ik)$ $(k \neq i)$; $I(ij; jj)$ $(i \neq j)$; $I(ij; jh)$, $I(ij; ik)$, $I(ij; jk)$ $(i \neq j;\ h, k \neq i, j)$ are represented by (2.7); (2.8); (2.10), respectively. By the way, it is noticed that $I(ij; ih)$ is symmetric with respect to suffices i and j, i.e., $I(ij; ih) = I(ij; jh)$ provided $h \neq i, j$.

Corresponding to (2.26) and (2.27) of VII, we may obtain partial sums of probabilities concerning various combinations of mother and first child. First, by summing up the probabilities (2.6) with respect to the suffix i, we get the partial sum concerning pairs of the same homozygote:

$$\sum_{i=1}^{m} I(ii; ii) = S_3 - S_4 - S_2 S_3 + \tfrac{1}{2} S_5 + \tfrac{1}{2} S_3^2. \tag{2.11}$$

Similarly, we get in turn the following results:

$$\sum_{i=1}^{m} \sum_{h \neq i} I(ii; ih) = S_2 - S_3 - 2S_2^2 + S_4 + 2S_2 S_3 - \tfrac{1}{2} S_5 - \tfrac{1}{2} S_3^2; \tag{2.12}$$

$$(2.13)\quad \sum_{i,j}{}'(I(ij;ii)+I(ij;jj))$$
$$=S_2-2S_3-\tfrac{3}{2}S_2^2+2S_4+\tfrac{5}{4}S_2S_3-\tfrac{1}{4}S_5+\tfrac{1}{4}S_3^2+\tfrac{3}{4}S_2S_4-\tfrac{3}{2}S_6,$$

$$(2.14)\quad \sum_{i,j}{}'I(ij;ij)$$
$$=S_2-2S_3-\tfrac{3}{2}S_2^2+2S_4+\tfrac{5}{4}S_2S_3-\tfrac{1}{4}S_5+\tfrac{1}{4}S_3^2+\tfrac{3}{4}S_2S_4-\tfrac{3}{2}S_6,$$

$$(2.15)\quad \sum_{i,j}{}'\sum_{h\neq i,j}(I(ij;ih)+I(ij;jh))$$
$$=1-5S_2+5S_3+3S_2^2-2S_4-\tfrac{1}{2}S_2S_3-\tfrac{5}{2}S_5-\tfrac{1}{2}S_3^2-\tfrac{3}{2}S_2S_4+3S_6;$$

both sums (2.13) and (2.14) coinciding each other.

If the type of first child is further eliminated by summation, then the partial sums given in (2.16) and (2.19) of VII are reobtainded:

$$(2.16)\quad \begin{aligned} &I(ii;ii)+\sum_{h\neq i}I(ii;ih)=P(ii),\\ &I(ij;ii)+I(ij;jj)+I(ij;ij)+\sum_{h\neq i,j}(I(ij;ih)+I(ij;jh))=P(ij) \\ &\qquad\qquad (i\neq j). \end{aligned}$$

It will also be obvious that the total sum of (2.5) gives again the *whole probability* of proving non-paternity; namely,

$$(2.17)\quad I\equiv\sum_{i\leq j;\,h\leq k}I(ij;\,hk)=1-2S_2+S_3-2S_2^2+2S_4+3S_2S_3-3S_5.$$

Similar remarks, as stated at the end of § 2 of VII, apply here also. It will further be noticed that the expressions of partial sums contain the power-sums up to sixth degree, inclusive, while that of the whole probability does those up to fifth.

By the way, we state that the ratio defined by

$$(2.18)\quad I(ij;hk)/\pi(ij;hk)$$

represents the probability of proving non-paternity, a type of first child being indifferent, against second child accompanied by fixed types A_{ij} and A_{hk} of mother and first child.

The reader will be able to tabulate the results obtained in the present section.

In case where recessive genes are existent, the corresponding modifications will also lead to the results. The problem and the results mentioned above can also be generalized to case of mixed mother-child combinations.

3. Decomposition of the whole probability I with regard to types of children.

We now proceed to discuss the decomposition of the whole probability I in (2.17) into sub-probabilities with respect to pairs of children types, corresponding to a discussion performed in § 4 of VIII. Such a sub-probability is derived by summing up the quantities (2.3) over the possible types A_{ij} of mother. Thus, we denote by

$$(3.1)\quad H(hk,fg)=\sum_{i\leq j}P(ij;hk,fg)$$

the *sub-probability of proving non-paternity, being indifferent to first child, against second child* A_{fg} *when first child is of the type* A_{hk}.

In case where children are both of the same homozygote A_{ff}, the possible types of mother being those containing the gene A_f at least one, we have the expression

$$(3.2)\quad H(ff,ff)=P(ff;ff,ff)+\sum_{j\neq f}P(jf;ff,ff)=\tfrac{1}{4}p_f^2(1+p_f)^2(1-p_f)^2.$$

Next, in case where both children are homozygotic but of different types, the possible type of mother being necessarily the homozygote containing genes in common with each of them, we get

$$(3.3)\qquad H(hh,ff)=P(hf;hh,ff)=\tfrac{1}{4}p_h^2p_f^2(1-p_f)^2 \qquad (h\neq f).$$

Similarly, we obtain in turn the following results:

$$(3.4)\quad \begin{aligned} H(hf,ff)&=P(ff;hf,ff)+P(hf;hf,ff)+\sum_{j\neq h,f}P(jf;hf,ff)\\ &=\tfrac{1}{2}p_hp_f^2(1+p_f)(1-p_f)^2 \qquad (h\neq f), \end{aligned}$$

$$(3.5)\quad \begin{aligned} H(hk,ff)&=P(hf;hk,ff)+P(kf;hk,ff)\\ &=\tfrac{1}{2}p_hp_kp_f^2(1-p_f)^2 \qquad (h,k\neq f;\ h\neq k); \end{aligned}$$

$$(3.6)\quad \begin{aligned} H(ff,fg)&=P(ff;ff,fg)+P(fg;ff,fg)+\sum_{j\neq f,g}P(jf;ff,fg)\\ &=\tfrac{1}{4}p_f^2p_g((1+p_f+p_g)(1-p_f-p_g)^2+(1+p_f-p_g)(1-p_g)^2)\\ &\qquad (f\neq g), \end{aligned}$$

$$(3.7)\quad \begin{aligned} H(fg,fg)&=P(ff;fg,fg)+P(gg;fg,fg)+P(fg;fg,fg)\\ &\quad+\sum_{j\neq f,g}(P(jf;fg,fg)+P(jg;fg,fg))\\ &=\tfrac{1}{4}p_fp_g(2-(p_f^2+p_g^2)-4p_fp_g+p_f^3+p_g^3-4p_fp_g(p_f+p_g)\\ &\quad+5p_fp_g(p_f^2+p_g^2)+6p_f^2p_g^2) \qquad (f\neq g), \end{aligned}$$

$$(3.8)\quad \begin{aligned} H(hf,fg)&=P(ff;hf,fg)+P(hf;hf,fg)\\ &\quad+P(fg;hf,fg)+P(hg;hf,fg)+\sum_{j\neq h,f,g}P(jf;hf,fg)\\ &=\tfrac{1}{4}p_hp_fp_g(2+2(p_f-p_g)-(3p_f^2-p_g^2)-8p_fp_g\\ &\quad+2p_f^3+p_fp_g(3p_f+5p_g)) \qquad (f\neq g;\ h\neq f,\ g), \end{aligned}$$

$$(3.9)\quad \begin{aligned} H(hh,fg)&=P(hf;hh,fg)+P(hg;hh,fg)\\ &=\tfrac{1}{4}p_h^2p_fp_g(2-2(p_f+p_g)+p_f^2+p_g^2) \qquad (f\neq g;\ h\neq f,\ g), \end{aligned}$$

$$(3.10)\quad \begin{aligned} H(hk,fg)&=P(hf;hk,fg)+P(hg;hk,fg)\\ &\quad+P(kf;hk,fg)+P(kg;hk,fg)\\ &=\tfrac{1}{2}p_hp_kp_fp_g(2-2(p_f+p_g)+p_f^2+p_g^2)\\ &\qquad (f\neq g;\ h,k\neq f,\ g;\ h\neq k). \end{aligned}$$

Thus, all the cases have essentially been worked out.

The quantity (3.2) represents the sub-probability of proving non-paternity, indifferent to first child, against second child when first child is of the same homozygote A_{ff} as that of second. On the other hand, if we sum up the quantities (3.3) with respect to h ($\neq f$), we get the sub-probability when first child is one of homozygotes different from that, A_{ff} say, of second; we thus obtain

$$(3.11)\qquad \sum_{h\neq f} H(hh,ff)=\tfrac{1}{4}p_f^2(1-p_f)^2(S_2-p_f^2).$$

Similar sub-probabilities are obtained as follows:

$$(3.12)\qquad \sum_{h\neq f} H(hf,ff)=\tfrac{1}{2}p_f^2(1+p_f)(1-p_f)^3,$$

$$(3.13)\qquad \sum_{h,k\neq f}{}' H(hk,ff)=\tfrac{1}{4}p_f^2(1-p_f)^2(1-S_2-2p_f+2p_f^2),$$

we further get, for $f\neq g$, the expressions

$$(3.14)\quad \sum_{h\neq f,g}(H(hf,fg)+H(hg,fg))=\tfrac{1}{4}p_fp_g(4-2(p_f^2+p_g^2)-16p_fp_g + 2(p_f^3+p_g^3)+8p_fp_g(p_f+p_g))(1-p_f-p_g),$$

$$(3.15)\quad \sum_{h\neq f,g} H(hh,fg)=\tfrac{1}{4}p_fp_g(2-2(p_f+p_g)+p_f^2+p_g^2)(S_2-p_f^2-p_g^2),$$

$$(3.16)\quad \sum_{h,k\neq f,g}{}' H(hk,fg)=\tfrac{1}{4}p_fp_g(2-2(p_f+p_g) +p_f^2+p_g^2)((1-p_f-p_g)^2-(S_2-p_f^2-p_g^2)).$$

The sum of (3.2) and of (3.11) to (3.13) yields the probability of proving non-paternity, indifferent to first child, against second child of homozygote A_{ff}; namely,

$$(3.17)\quad H(ff,ff)+\sum_{h\neq f}(H(hh,ff)+H(hf,ff))+\sum_{h,k\neq f}{}' H(hk,ff) =p_f^2(1-p_f)^2=R(ff).$$

Similarly, the sum of $H(ff,fg)$ in (3.6) and of $H(gg,fg)$ analogously obtainable and of (3.7) and of (3.14) to (3.16) yields the probability against second child of heterozygote A_{fg} $(f\neq g)$; namely, we get, for $f\neq g$,

$$(3.18)\quad H(ff,fg)+H(gg,fg)+H(fg,fg) +\sum_{h\neq f,g}(H(hf,fg)+H(hg,fg)+H(hh,fg))+\sum_{h,k\neq f,g}{}' H(hk,fg) =p_fp_g(2-2(p_f+p_g)+p_f^2+p_g^2-4p_fp_g+3p_fp_g(p_f+p_g))=R(fg).$$

The results just obtained in (3.17) and (3.18) coincide, of course, with those in (3.2) and (3.3) of VIII, respectively.

We have derived in the last two expressions the probabilities corresponding to types of second child. Analogous probabilities may be derived corresponding to types of first child. The probability of proving non-paternity, first child being indifferent, against second child accompanied by first child of homozygote A_{hh} is given by

$$(3.19)\quad H(hh,hh)+\sum_{f\neq h}(H(hh,ff)+H(hh,hf))+\sum_{f,g\neq h}{}' H(hh,fg) =\tfrac{1}{4}p_h^2(4-6S_2+3S_3-(4+2S_2-3S_3)p_h+(4+3S_2)p_h^2+p_h^3-6p_h^4),$$

and that accompanied by first child of heterozygote A_{hk} $(h\neq k)$ by

$$(3.20)\quad H(hk,hh)+H(hk,kk)+H(hk,hk) +\sum_{f\neq h,k}(H(hk,hf)+H(hk,kf)+H(hk,ff))+\sum_{f,g\neq h,k}{}' H(hk,fg) =\tfrac{1}{4}p_hp_k(8-8S_2+4S_3-(4+4S_2-3S_3)(p_h+p_k)+(2+3S_2)(p_h^2+p_k^2) -4p_hp_k+4(p_h^3+p_k^3)+3p_hp_k(p_h+p_k)-6(p_h^4+p_k^4)).$$

The results obtained in the present section can also be generalized to mixed case corresponding to (2.20), but the detailed discussion will be left to the reader. *— To be continued —*