60. Probability-theoretic Investigations on Inheritance. $\mathfrak{X}_2$. Non-Paternity Concerning Mother-Child-Child Combinations.

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3. Probability a posteriori of a father against a mother-child combination.

A new essential problem, being characteristic with respect to mother-child-child combination, will arise; i. e., given a mother-child-child combination, at how much rate a father of first child can assert his non-paternity against second child? In the problem discussed in § 1, the whole of men except a father of second child having been taken into account against a given mother-child-child combination, the relation to first child has not been directly necessary to be considered, and hence the use has been made of the quantities (1.1) consisting of the V's concerning general distribution-frequencies. In the present problem, however, the object in question being restricted to a father of first child, the possible types of him are limited according to mother-child-child combinations, and hence the V's in (1.1) must be replaced by probabilities a posteriori of a father for combinations of mother and her first child.

The probabilities a posteriori in question can be estimated by means of Bayes' theorem on probability of causes referred to at the end of § 1 in IV. In fact, we may take, as probability a priori, the frequency of general distribution. On the other hand, the probability of an event that a father produce a child of each type with a mother of given type has been listed in a table in § 3 of I, a remark stated immediately subsequent to (1.8) of IV being here also to be remembered.

Now, in general, given a pair $(A_{ij}; A_{hk})$ of a mother and her child, the probability a posteriori of a father to be of type A_{ab} be denoted by

$$(3.1) \qquad Z(ab, ij; hk),$$

which will be explicitly determined in the following lines. Of course, only the cases are essential where at least a suffix among h, k coincides with a or b and with i or j; otherwise, the quantity (3.1) may by understood to be equal to zero.

We first consider a mother-child combination consisting of the same homozygote, $(A_{ii}; A_{ii})$ say. Only a man with at least one gene A_i can then be a father. The mating $A_{ii} \times A_{ii}$ produces a child A_{ii} alone, while the mating $A_{ik} \times A_{ii}$ $(k \neq i)$ produces a child A_{ii} with probability 1/2. Hence, the probability a posteriori of a father to be of type A_{ii} or A_{ib} $(b \neq i)$ is, in view of Bayes' theorem, given by

$$(3.2) \qquad \bar{A}_{ii}/(\bar{A}_{ii}+\sum_{k\neq i}\tfrac{1}{2}\bar{A}_{ik}), \qquad \tfrac{1}{2}\bar{A}_{ib}/(\bar{A}_{ii}+\sum_{k\neq i}\tfrac{1}{2}\bar{A}_{ik}),$$

respectively. The denominator common to both expressions in (3.2) is nothing but the probability of a child A_{ii} produced from a mother of fixed type A_{ii}, as already noticed in (1.27) of IV, namely

$$(3.3) \qquad \bar{A}_{ii}+\sum_{k\neq i}\tfrac{1}{2}\bar{A}_{ik}=\pi(ii;ii)/\bar{A}_{ii}.$$

A relation analogous to the last one will be valid also for every mother-child combination. Thus, in view of (3.3), expressions in (3.2) are written as follows:

$$(3.4) \qquad \begin{aligned} Z(ii,\ ii;\ ii) &= \bar{A}_{ii}^2/\pi(ii;\ ii)=p_i, \\ Z(ib,\ ii;\ ii) &= \tfrac{1}{2}\bar{A}_{ib}\bar{A}_{ii}/\pi(ii;\ ii)=p_b \qquad (b\neq i), \end{aligned}$$

respectively. Similarly, the following results will be derived:

$$(3.5) \qquad \begin{aligned} Z(hh,ii;ih) &= \bar{A}_{hh}\bar{A}_{ii}/\pi(ii;ih)=p_h, \\ Z(hb,ii;ih) &= \tfrac{1}{2}\bar{A}_{hb}\bar{A}_{ii}/\pi(ii;ih)=p_b \qquad (b\neq h); \end{aligned}$$

$$(3.6) \qquad \begin{aligned} Z(ii,\ ij;\ ii) &= \tfrac{1}{2}\bar{A}_{ii}\bar{A}_{ij}/\pi(ij;\ ii)=p_i \qquad (i\neq j), \\ Z(ib,\ ij;\ ii) &= \tfrac{1}{4}\bar{A}_{ib}\bar{A}_{ij}/\pi(ij;\ ii)=p_b \qquad (i\neq j;\ b\neq i); \end{aligned}$$

$$(3.7) \qquad \begin{aligned} Z(ii,\ ij;\ ij) &= \tfrac{1}{2}\bar{A}_{ii}\bar{A}_{ij}/\pi(ij;\ ij)=p_i^2/(p_i+p_j) \qquad (i\neq j), \\ Z(ij,\ ij;\ ij) &= \tfrac{1}{2}\bar{A}_{ij}^2/\pi(ij;\ ij)=2p_ip_j/(p_i+p_j) \qquad (i\neq j), \\ Z(ib,\ ij;\ ij) &= \tfrac{1}{4}\bar{A}_{ib}\bar{A}_{ij}/\pi(ij;\ ij)=p_ip_b/(p_i+p_j) \qquad (i\neq j;\ b\neq i,j); \end{aligned}$$

$$(3.8) \qquad \begin{aligned} Z(hh,ij;ih) &= \tfrac{1}{2}\bar{A}_{hh}\bar{A}_{ij}/\pi(ij;ih)=p_h \qquad (h\neq i,j), \\ Z(hb,ij;ih) &= \tfrac{1}{4}\bar{A}_{hb}\bar{A}_{ij}/\pi(ij;ih)=p_b \qquad (h\neq i,j;\ b\neq h). \end{aligned}$$

The results obtained in (3.4) to (3.8) may be listed as follows; different suffices denoting different genes and use being made of an abbreviation

$$(3.9) \qquad \varepsilon_{ij}=1/(p_i+p_j).$$

Mother	Child \ Father	A_{ii}	A_{ib}	A_{hh}	A_{ih}	A_{bh}
A_{ii}	A_{ii}	p_i	p_b	0	p_h	0
	A_{ih}	0	0	p_h	p_i	p_b

Mother	Child \ Father	A_{ii}	A_{jj}	A_{ij}	A_{ib}	A_{jb}	A_{hh}	A_{ih}	A_{jh}	A_{bh}
A_{ij}	A_{ii}	p_i	0	p_j	p_b	0	0	p_h	0	0
	A_{jj}	0	p_j	p_i	0	p_b	0	0	p_h	0
	A_{ij}	$p_i^2\varepsilon_{ij}$	$p_j^2\varepsilon_{ij}$	$2p_ip_j\varepsilon_{ij}$	$p_ip_b\varepsilon_{ij}$	$p_jp_b\varepsilon_{ij}$	0	$p_ip_h\varepsilon_{ij}$	$p_jp_h\varepsilon_{ij}$	0
	A_{ih} or A_{jh}	0	0	0	0	0	p_h	p_i	p_j	p_b

4. Non-paternity of a father of a child against another child.

The quantity (3.1) denoting the probability a posteriori of a combination has thus been determinined. Accordingly, the problem stated at the beginning of § 3 can then be discussed similarly as before. Now, given a pair of a mother and her second child, possible types of a man being not a father of second child were already listed in a table of § 3 in VIII, among which the types possible for a father of first child are merely to be considered for the present purpose.

Let a mother-child-child combination $(A_{ij}; A_{hk}, A_{fg})$ be presented. Then the probability of an event that a father of its first child can prove his non-paternity, based upon an inherited character under consideration, against second child be denoted by

$$(4.1) \qquad V_0(ij; hk, fg).$$

The probability of a composed event that such a combination is presented and non-paternity proof is possible is then expressed by

$$(4.2) \qquad X(ij; hk, fg) \equiv \pi_0(ij; hk, fg)\, V_0(ij; hk, fg),$$

a quantity fundamental in the present discussion.

A remarkable fact would be previously noticed. In fact, if the type of second child coincides with that of first child, then any type of father of first child can also be a type possible for father of second child, whence the quantity (4.1) does vanish; namely, an identical relation holds good:

$$(4.3) \qquad V_0(ij; hk, hk) = 0.$$

In order to determine explicit expressions for (4.1), we first consider a case where a mother is of a homozygote. In view of a remark just stated in (4.3), we get

$$(4.4) \qquad V_0(ii; ii, ii) = 0, \qquad V_0(ii; ih, ih) = 0 \qquad (h \neq i).$$

With respect to combinations containing different types of first and second children, we get

$$(4.5) \qquad V_0(ii; ii, ih) = 1 - Z(ih, ii; ii) = 1 - p_h \qquad (h \neq i),$$

$$(4.6) \qquad V_0(ii; ih, ii) = 1 - Z(ih, ii; ih) = 1 - p_i \qquad (h \neq i),$$

$$(4.7) \qquad V_0(ii; ih, ik) = 1 - Z(hk, ii; ih) = 1 - p_k \quad (h, k \neq i;\ h \neq k),$$

since, in (4.5) and (4.6) or in (4.7), any type of father of first child except A_{ih} or A_{hk} respectively can not be a type of father of second child.

We next consider a mother of a heterozygote. Here also the relation (4.3) remains valid. Against mother-child-child combination $(A_{ij}; A_{ii}, A_{jj})$ $(i \neq j)$ any type of father of first child except A_{ij} must be excluded, whence follows the relation

(4.8) $$V_0(ij;\, ii,\, jj)=1-Z(ij,\, ij;\, ii)=1-p_j \qquad (i\neq j).$$

Against $(A_{ij};\, A_{ii},\, A_{ij})$, no type can be excluded. In fact, a father of first child A_{ii} must contain at least one gene A_i and hence can produce with mother a child A_{ij}. Thus, we have

(4.9) $$V_0(ij;\, ii,\, ij)=0.$$

In similar ways, we determine the probabilities in question as follows:

(4.10) $$V_0(ij;\, ii,\, ih)=1-Z(ih,\, ij;\, ii)=1-p_h \qquad (i\neq j;\, h\neq i,\, j),$$

(4.11) $$V_0(ij;\, ii,\, jh)=1-Z(ih,\, ij;\, ii)=1-p_h \qquad (i\neq j;\, h\neq i,\, j),$$

(4.12) $$V_0(ij;\, ij,\, ii)=Z(jj,\, ij;\, ij)+\sum_{b\neq i,j} Z(jb,\, ij;\, ij) = p_j(1-p_i)/(p_i+p_j) \qquad (i\neq j),$$

(4.13) $$V_0(ij;\, ij,\, ih)=1-Z(ih,\, ij;\, ij)-Z(jh,\, ij;\, ij) =1-p_h \qquad (i\neq j;\, h\neq i,\, j),$$

(4.14) $$V_0(ij;\, ih,\, ii)=1-Z(ih,\, ij;\, ih)=1-p_i \qquad (i\neq j;\, h\neq i,\, j),$$

(4.15) $$V_0(ij;\, ih,\, jj)=1-Z(jh,\, ij;\, ih)=1-p_j \qquad (i\neq j;\, h\neq i,\, j),$$

(4.16) $$V_0(ij;\, ih,\, ij)=1-Z(ih,\, ij;\, ih)-Z(jh,\, ij;\, ih) =1-p_i-p_j \qquad (i\neq j;\, h\neq i,\, j),$$

(4.17) $$V_0(ij;\, ih,\, ik)=1-Z(hk,\, ij;\, ih)=1-p_k \qquad (i\neq j;\, h,\, k\neq i,\, j;\, h\neq k),$$

(4.18) $$V_0(ij;\, ih,\, jh)=0 \qquad (i\neq j;\, h\neq i,\, j),$$

(4.19) $$V_0(ij;\, ih,\, jk)=1-Z(hk,\, ij;\, ih)=1-p_k \quad (i\neq j;\, h,k\neq i,\, j;\, h\neq k).$$

Remembering the identity (4.3), all the possible cases have thus essentially been worked out.

Probabilities of mother-child-child combinations, i. e., the π_0's, having already been determined in § 5 of IV, we can immediately caculate the desired probabilities defined in (4.2) in concrete forms.

If we eliminate the type of first child by summing up the quantities (4.2) over all the possible indices h, k, then the corresponding partial probability of proving non-paternity concerining one-child family discussed in VII is obtained; in other words, we shall get

(4.20) $$\sum_{h\leq k} X(ij;\, hk,\, fg)=P(ij;\, fg),$$

a relation which can also immediately be verified by direct calculation; denoting, of course, the quantity introduced in (2.2) of VII.

However, if we eliminate the type of second child by summing up the quantities over all the possible indices f, g, then a partial probability of a new sort will be obtained. We introduce a notation

(4.21) $$L(ij;hk)=\sum_{f\leq g} X(ij;hk,fg),$$

which represents the probability of an event that, given a mother A_{ij} and her first child A_{hk}, *the father of the first child* can assert his non-paternity against her second child produced with another man. Calculating (4.21) in concrete form, we get the following results:

(4.22) $$L(ii;\ ii)=\sum_{k\neq i} X(ii;ii,\ ik)=p_i^3(1-S_2-p_i+p_i^2),$$

(4.23) $$L(ii;ih)=\sum_{k\neq h} X(ii;ih,\ ik)=p_i^2p_h(1-S_2-p_h+p_h^2) \qquad (h\neq i);$$

(4.24) $$\begin{aligned}L(ij;\ ii)&=X(ij;\ ii,jj)+\sum_{k\neq i,j}(X(ij;ii,\ ik)+X(ij;ii,jk))\\&=p_i^2p_j(1-S_2-(p_i+\tfrac{1}{2}p_j)+p_i^2+\tfrac{1}{2}p_j^2) \qquad (i\neq j),\end{aligned}$$

(4.25) $$\begin{aligned}L(ij;\ ij)&=X(ij;ij,\ ii)+X(ij;ij,jj)+\sum_{k\neq i,j}(X(ij;ij,ik)+X(ij;ij,jk))\\&=p_ip_j((1-S_2)(p_i+p_j)-(p_i^2+p_j^2)-p_ip_j\\&\qquad+(p_i^3+p_j^3)+\tfrac{1}{2}p_ip_j(p_i+p_j)) \qquad (i\neq j),\end{aligned}$$

(4.26) $$\begin{aligned}L(ij;ih)&=X(ij;ih,\ ij)+\sum_{k\neq j,h} X(ij;ih,\ ik)+\sum_{k\neq i,h} X(ij;ih,jk)\\&=p_ip_jp_h(1-S_2-p_ip_j-p_h+p_h^2) \qquad (i\neq j;h\neq i,j).\end{aligned}$$

By the way, we further calculate partial sums of probabilities according to various pairs of mother and first child. The results are as follows:

(4.27) $$\sum_{i=1}^{m} L(ii;\ ii)=S_3-S_4-S_2S_3+S_5,$$

(4.28) $$\sum_{i=1}^{m}\sum_{h\neq i} L(ii;ih)=S_2-S_3-2S_2^2+S_4+2S_2S_3-S_5;$$

(4.29) $$\sum_{i,j}{}'\ (L(ij;\ ii)+L(ij;\ jj))=S_2-2S_3-\tfrac{3}{2}S_2^2+\tfrac{5}{2}S_4+\tfrac{3}{2}S_2S_3-\tfrac{3}{2}S_5,$$

(4.30) $$\sum_{i,j}{}'\ L(ij;ij)=S_2-2S_3-\tfrac{3}{2}S_2^2+\tfrac{5}{2}S_4+\tfrac{3}{2}S_2S_3-\tfrac{3}{2}S_5,$$

(4.31) $$\sum_{i,j}{}'\sum_{h\neq i,j}(L(ij;ih)+L(ij;jh))=1-5S_2+5S_3+3S_2^2-3S_4-S_2S_3.$$

If we further sum up the quantities (4.27) to (4.28) and (4.29) to (4.31), according to mothers of homozygotes and of heterozygotes respectively, then we get

(4.32) $$S_2(1-2S_2+S_3), \qquad 1-3S_2+S_3+2S_4+2S_2S_3-3S_5,$$

while the sum of the last two expressions in (4.32) implies

(4.33) $$L=1-2S_2+S_3-2S_2^2+2S_4+2S_2S_3-3S_5,$$

which represents just the whole probability for one-child case already mentioned in (2.20) of VII and (2.17) of IX; that is,

(4.34) $$L=P=J.$$

—To be continued—