71. Probability-theoretic Investigations on Inheritance. XI_1. Absolute Non-Paternity.

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1. Absolute non-paternity.

In several preceding chapters[1] we have discussed the problems of proving non-paternity exclusively on the supposition that an inherited character of mother has been known as well as that of her child or those of her children. A fundamental postulate has accordingly based on a fact that any type not able to produce with her a child in question can never belong to a true father. Hence, the problems are, so to speak, those relative to a type of mother.

On the other hand, there are cases where non-paternity proof is possible *without* taking a type of mother into account; namely, there exist pairs of types which can never belong to father and his any child. Non-paternity will then be established *absolutely*, i.e., with no regard to a type of mother.

If non-paternity proof is possible absolutely, then it is, of course, also possible relatively to a type of mother. Hence, any probability of proving absolute non-paternity does never exceed the corresponding one of proving relative non-paternity.

We now begin with a problem stating that: *Given a child and a man, at how many rate the non-paternity proof can be absolutely established?* Let a type of a child be A_{ij} and that of a man be A_{hk}. The absolute non-paternity can be verified if and only if there exists no common suffix between i, j and h, k. Since it is the same to

1) Y. Komatu, Probability-theoretic investigations on inheritance. I. Distribution of genes; II. Cross-breeding phenomena; III. Further discussions on cross-breeding; IV. Mother-child combinations; V. Brethren-combinations; VI. Rate of danger in random blood transfusion; VII. Non-paternity problems; VIII. Further discussions on non-paternity; IX. Non-paternity problems concerning mother-children combinations; X. Non-paternity problems concerning mother-child-child combinations. Proc. Jap. Acad. **27** (1951), I. 371-377; II. 378-383, 384-387; III. 459-465, 466-477, 478-483; IV. 587-592, 593-597, 598-603, 605-610, 611-614, 615-620; V. 689-693, 694-699; **28** (1952), VI. 54-58. VII. 102-104, 105-108, 109-111, 112-115, 116-120, 121-125; VIII. 162-164, 165-168, 169-171; IX. 207-212, 213-217, 218-223, 224-229; X. 249-253, 254-258, 259-264. These papers will be referred to as I; II; III; IV; V; VI; VII; VIII; IX; X.

us whether a type of a child or of a man is considered to be basic, we shall here classify the cases according to the former and thus denote by

$$(1.1) \qquad C(ij)$$

the probability of an event that a child of A_{ij} appears and then the proof of absolute non-paternity can be verified.

Against a homozygotic child A_{ii}, anyone not possessing a gene A_i, and such one alone, can never be a true father, what implies

$$(1.2) \qquad C(ii)=p_i^2(1-p_i)^2.$$

Against a heterozygotic child $A_{ij}(i\neq j)$, anyone possessing neither a gene A_i nor A_j, and such one alone, can never be a true father, what implies

$$(1.3) \qquad C(ij)=2p_ip_j(1-p_i-p_j)^2 \qquad (i\neq j).$$

The partial probability of proving absolute non-paternity over homozygotic children is given by

$$(1.4) \qquad \sum_{i=1}^{m} C(ii)=S_2-2S_3+S_4,$$

and that over heterozygotic children by

$$(1.5) \qquad \sum_{i,j}{}' C(ij)=1-5S_2+6S_3+2S_2^2-4S_4.$$

The sum of the last two quantities yields the *whole probability of proving absolute non-paternity* expressed in the form

$$(1.6) \qquad C=1-4S_2+4S_3+2S_2^2-3S_4.$$

In an individual case of inherited character where recessive genes may be existent, the pairs without father-child relationship can immediately be read from a table in §1 of IV. In fact, a table concerning mother-child combinations may, as it is, also be regarded as the one concerning father-child combinations, so that the pairs with the vanishing probability are just those incompatible as a father and his child. Thus, for instance, in case of ABO blood type, O and AB being incompatible with AB and O respectively, and these alone being incompatible pairs, the whole probability of proving absolute non-paternity is equal to

$$(1.7) \qquad C_{ABO} = 4pqr^2.$$

In case of A_1A_2BO blood type, A_2 and A_1B becoming moreover incompatible each other, the whole probability is then given by

$$(1.8) \qquad C_{A_1A_2BO}=4pqr^2+4p_1p_2q(p_2+2r).$$

The case of MN blood type is a special one ($m=2$) of general discussion; the whole probability is given by

(1.9) $$C_{MN} = 2s^2t^2.$$

In case of Q or $Qq_{\pm}$ blood type, any pair being compatible, the whole probability vanishes, i.e.,

(1.10) $$C_Q = C_{Qq\pm} = 0.$$

Generalization to mixed combinations is also possible. The results are as follows; distributions of populations to which a mother of child and a man in question belong being denoted by $\{p_i\}$ and $\{p'_i\}$, respectively:

(1.11) $$C'(ii) = p_i p'_i (1-p'_i)^2,$$

(1.12) $$C'(ij) = (p_i p'_j + p_j p'_i)(1-p'_i-p'_j)^2 \qquad (i \neq j);$$

(1.13) $$\sum_{i=1}^{m} C'(ii) = S_{1,1} - 2S_{1,2} + S_{1,3},$$

(1.14) $$\sum_{i,j}' C'(ij) = 1 - (2S'_2 + 3S_{1,1}) + (S'_3 + 5S_{1,2}) + 2S'_2 S_{1,1} - 4S_{1,3};$$

the *whole probability* of proving absolute non-paternity being then

(1.15) $$C' = 1 - 2(S'_2 + S_{1,1}) + (S'_3 + 3S_{1,2}) + 2S'_2 S_{1,1} - 3S_{1,3}.$$

In concrete cases the results become as follows:

(1.16) $$C'_{ABO} = 2rp'q'r' + (pq' + qp')r'^2,$$

(1.17) $$C'_{A_1A_2BO} = 2rp'q'r' + (pq' + qp')r'^2 + (p_1q' + qp'_1)p'_2(p'_2 + 2r') + 2(p_2p'_2 + p_2r' + rp'_2)p'_1q',$$

(1.18) $$C'_{MN} = s't'(st' + ts'),$$

(1.19) $$C'_Q = C'_{Qq\pm} = 0.$$

2. Absolute non-paternity against children of the same family.

Probabilities of brethren combinations consisting of children of the same family have already been listed in a table in § 1 of V. We next denote by

(2.1) $$D(ij,\ hk)$$

the probability of an event that such a brethren combination (A_{ij}, A_{hk}) *appears and then the proof of absolute non-paternity can be established against both of them.*

The present discussions correspond to those in § 4 of IX; namely, the problem concerns the absolute non-paternity against both children of the same family *separately*, i.e., against both first and second children indifferent to second and first children respectively. The discussions corresponding to those in § 7 of IX will be performed in a subsequent section.

Now, against a combination (A_{ii}, A_{ii}) anyone not possessing A_i (and, of course, such alone) can never be a true father, what implies

$$(2.2)\qquad D(ii,\, ii)=\sigma(ii,\, ii)(1-p_i)^2=\tfrac{1}{4}p_i^2(1+p_i)^2(1-p_i)^2.$$

Against a combination $(A_{ii},\, A_{hh})\,(h\neq i)$ anyone possessing neither A_i nor A_h can never be a true father, what implies

$$(2.3)\qquad D(ii,\, hh)=\sigma(ii,\, hh)\,(1-p_i-p_h)^2=\tfrac{1}{4}p_i^2p_h^2(1-p_i-p_h)^2 \qquad (h\neq i).$$

Similar considerations lead to the following results:

$$(2.4)\qquad D(ii,\, ih)=\tfrac{1}{2}p_i^2p_h(1+p_i)(1-p_i-p_h)^2 \qquad (h\neq i),$$

$$(2.5)\qquad D(ii,\, hk)=\tfrac{1}{2}p_i^2p_hp_k(1-p_i-p_h-p_k)^2 \qquad (h,\, k\neq j;\ h\neq k);$$

$$(2.6)\qquad D(ij,\, ij)=\tfrac{1}{2}p_ip_j(1+p_i+p_j+2p_ip_j)(1-p_i-p_j)^2 \qquad (i\neq j),$$

$$(2.7)\qquad D(ij,\, ih)=\tfrac{1}{2}p_ip_jp_h(1+2p_i)(1-p_i-p_j-p_h)^2 \qquad (i\neq j;\ h\neq i,\, j),$$

$$(2.8)\qquad D(ij,\, hk)=p_ip_jp_hp_k(1-p_i-p_j-p_h-p_k)^2, \qquad (i\neq j;\ h,\, k\neq i,\, j;\ h\neq k).$$

Remembering an evident symmetry relation

$$(2.9)\qquad D(ij,\, hk)=D(hk,\, ij) \qquad (i,\, j,\, h,\, k=1,\, \ldots,\, m),$$

all the possible cases have thus essentially been worked out.

Partial sums of probabilities with respect to type of first child can be obtained in a usual manner, yielding

$$(2.10)\qquad \begin{aligned} D(ii)&\equiv D(ii,\, ii)+\sum_{h\neq i}(D(ii,\, ih)+D(ii,\, hh))+\sum_{h,\, k\neq i}{}' D(ii,\, hk)\\ &=p_i^2(1-2S_1+\tfrac{3}{2}S_3+\tfrac{1}{2}S_2^2-\tfrac{3}{4}S_4\\ &\quad -(2-2S_2+\tfrac{1}{2}S_3)p_i+(3-S_2)p_i^2-\tfrac{7}{2}p_i^3+\tfrac{7}{4}p_i^4), \end{aligned}$$

$$(2.11)\qquad \begin{aligned} D(ij)&\equiv D(ij,\, ii)+D(ij,\, jj)+D(ij,\, ij)\\ &\quad+\sum_{h\neq i,\, j}(D(ij,\, hh)+D(ij,\, ih)+D(ij,\, jh))+\sum_{h,\, k\neq i,\, j}{}' D(ij,\, hk)\\ &=2p_ip_j(1-2S_2+\tfrac{3}{2}S_3+\tfrac{1}{2}S_2^2-\tfrac{3}{4}S_4-(2-2S_2+\tfrac{1}{2}S_3)(p_i+p_j)\\ &\quad+(3-S_2)(p_i^2+p_j^2)+2p_ip_j-\tfrac{7}{2}(p_i^3+p_j^3)-2p_ip_j(p_i+p_j)\\ &\quad+\tfrac{7}{4}(p_i^4+p_j^4)+\tfrac{1}{2}p_ip_j(p_i^2+p_j^2)+p_i^2p_j^2) \qquad (i\neq j). \end{aligned}$$

Sub-probabilities over homozygotic and heterozygotic first children become, respectively.

$$(2.12)\qquad \begin{aligned} \sum_{i=1}^{m} D(ii)&=S_2-2S_3-2S_2^2+3S_4\\ &\quad+\tfrac{7}{2}S_2S_3-\tfrac{7}{2}S_5+\tfrac{1}{2}S_2^3-\tfrac{1}{2}S_3^2-\tfrac{7}{4}S_2S_4+\tfrac{7}{4}S_6, \end{aligned}$$

$$(2.13)\qquad \begin{aligned} \sum_{i,\, j}{}' D(ij)&=1-7S_2+\tfrac{23}{2}S_3+\tfrac{17}{2}S_2^2-\tfrac{63}{4}S_4\\ &\quad-\tfrac{25}{2}S_2S_3+\tfrac{29}{2}S_5-\tfrac{1}{2}S_2^3+2S_3^2+\tfrac{15}{4}S_2S_4-\tfrac{11}{2}S_6, \end{aligned}$$

the sum of which represents, of course, the *whole probability*

$$(2.14)\qquad \begin{aligned} D&=1-6S_2+\tfrac{19}{2}S_3+\tfrac{13}{2}S_2^2-\tfrac{51}{4}S_4\\ &\quad-9S_2S_3+11S_5+\tfrac{3}{2}S_3^2+2S_2S_4-\tfrac{15}{4}S_6. \end{aligned}$$

Generalization to mixed combinations is possible, while it will be omitted and left to the reader.

As illustrative examples we state here the whole probabilities against both children separately in case of ABO, Q, $Qq_\pm$ as well as MN blood types; in three former cases, the existence of recessive genes requires a careful consideration. The results are as follows:

$$D_{ABO}=\tfrac{1}{2}pqr^2(3+r+r^2+2pq), \tag{2.15}$$

$$D_Q=D_{Qp\pm}=0, \tag{2.16}$$

$$D_{MN}=\tfrac{1}{2}s^2t^2(5-2st). \tag{2.17}$$

3. Absolute non-paternity against a distinguished child at any rate or alone.

Probabilities of proving non-paternity against a distinguished child, for instance, a second say, among brethren of the same family can be discussed in a similar way corresponding to § 2 of IX. But, the detail will be left to the reader. We notice here only that the whole probability in this case will naturally coincide with C in (1.6). And hence, *given a brethren combination of the same family, the case where the proof of absolute non-paternity is possible against second alone possesses the whole probability*

$$\begin{aligned} C-D=&2S_2-\tfrac{11}{2}S_3-\tfrac{9}{2}S_2^2+\tfrac{39}{4}S_4\\ &+9S_2S_3-11S_5-\tfrac{3}{2}S_3^2-2S_2S_4+\tfrac{15}{4}S_6; \end{aligned} \tag{3.1}$$

the result corresponding to (6.3) of IX.

We now proceed to a more proper problem concerning brethren of the *same family*. Corresponding to § 7 of IX, if it is sure that both children belong to the same family, the non-paternity proof would be established against both children provided that it is established against at least one child among them. The whole probability of the present problem is immediately obtained. Namely, the same reason which has led to (7.5) of IX leads now to an expression

$$\begin{aligned} \tilde{D}\equiv 2C-D=&1-2S_2-\tfrac{3}{2}S_3-\tfrac{5}{2}S_2^2-\tfrac{27}{4}S_4\\ &+9S_2S_3-11S_5-\tfrac{3}{2}S_3^2-2S_2S_4+\tfrac{15}{4}S_6. \end{aligned} \tag{3.2}$$

As a basic quantity, we introduce, corresponding to (7.2) of IX or rather to $\tilde{J}(ij, hk)$ contained in (7.10) to (7.14) of IX, the probability of proving absolute non-paternity against at least one child among a brethren-combination of the same family (and hence simultaneously against both of them) which will be denoted by

$$\tilde{D}(ij, hk)\equiv\sigma(ij, hk)\left(\frac{C(ij)}{\bar{A}_{ij}}+\frac{C(hk)}{\bar{A}_{hk}}\right)-D(ij, hk); \tag{3.3}$$

the presented combination being (A_{ij}, A_{hk}). In view of the definition (3.3), the relation

$$\tilde{D} \equiv \sum \tilde{D}(ij,\ hk) = 2C - D \tag{3.4}$$

is obvious. Explicit expressions will be derived, corresponding to (2.2) to (2.8), as follows:

$$\tilde{D}(ii,\ ii) = \tfrac{1}{4}p_i^2(1+p_i)^2(1-p_i)^2, \tag{3.5}$$

$$\tilde{D}(ii,\ hh) = \tfrac{1}{4}p_i^2p_h^2(1-2p_ip_h) \qquad (h \neq i), \tag{3.6}$$

$$\tilde{D}(ii,\ ih) = \tfrac{1}{2}p_i^2p_h(1+p_i)(1-p_i)^2 \qquad (h \neq i), \tag{3.7}$$

$$\tilde{D}(ii,\ hk) = \tfrac{1}{2}p_i^2p_hp_k(1-2p_i(p_h+p_k)) \qquad (h,\ k \neq i;\ h \neq k); \tag{3.8}$$

$$\tilde{D}(ij,\ ij) = \tfrac{1}{2}p_ip_j(1+p_i+p_j+2p_ip_j)(1-p_i-p_j)^2 \qquad (i \neq j), \tag{3.9}$$

$$\tilde{D}(ij,\ ih) = \tfrac{1}{2}p_ip_jp_h(1+2p_j)((1-p_i)^2-2p_jp_h) \quad (i \neq j;\ h \neq i,\ j), \tag{3.10}$$

$$\tilde{D}(ij,\ hk) = p_ip_jp_hp_k(1-2(p_i+p_j)(p_h+p_k)) \qquad (i \neq j;\ h,\ k \neq i,\ j;\ h \neq k). \tag{3.11}$$

Partial sums corresponding to (2.10) and (2.11) become, respectively,

$$\tilde{D}(ii) = p_i^2(1-(\tfrac{3}{2}+S_2-\tfrac{1}{2}S_3)p_i-(\tfrac{1}{4}-S_2)p_i^2+2p_i^3-\tfrac{7}{4}p_i^4), \tag{3.12}$$

$$\begin{aligned}\tilde{D}(ij) = 2p_ip_j(1 &- (1+\tfrac{3}{2}S_2-\tfrac{1}{2}S_3)(p_i+p_j)-(1-S_2)(p_i^2+p_j^2)+\tfrac{1}{2}p_ip_j \\ &+ \tfrac{11}{4}(p_i^3+p_j^3)+2p_ip_j(p_i+p_j) \\ &- \tfrac{7}{4}(p_i^4+p_j^4)-\tfrac{1}{2}p_ip_j(p_i^2+p_j^2)-p_i^2p_j^2) \qquad (i \neq j).\end{aligned} \tag{3.13}$$

Hence, we get, corresponding to (2.12), (2.13) and (2.14),

$$\sum_{i=1}^{m} \tilde{D}(ii) = S_2-\tfrac{3}{2}S_3-\tfrac{1}{4}S_4-S_2S_3+2S_5+\tfrac{1}{2}S_3^2+S_2S_4-\tfrac{7}{4}S_6, \tag{3.14}$$

$$\begin{aligned}\sum_{i,\,j}{}' \tilde{D}(ij) = 1-3S_2-\tfrac{5}{2}S_2^2+7S_4+10S_2S_3 \\ -13S_5-2S_3^2-3S_2S_4+\tfrac{11}{2}S_6;\end{aligned} \tag{3.15}$$

$$\begin{aligned}\tilde{D} = 1-2S_2-\tfrac{3}{2}S_3-\tfrac{5}{2}S_2^2+\tfrac{27}{4}S_4 \\ +9S_2S_3-11S_5-\tfrac{3}{2}S_3^2-2S_2S_4+\tfrac{15}{4}S_6.\end{aligned} \tag{3.16}$$

The last result is nothing but the one already stated in (3.2).

—To be continued—