74. On a Theorem of K. Yosida.

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1. A topic in the theory of partial differential equations which has received attention of recent years is the question of the behaviour at infinity of solutions which satisfy a null condition on the interior boundary of an infinite region, but which do not vanish identically. For the ordinary wave-equation we have the radiation condition of Sommerfeld, with its electromagnetic analogue. Analogous results have also been established for parabolic equations. Recently K. Yosida (Proc. Japan Acad., 27, 214–215 (1951)) has considered the equation

$$\Delta h(x)=m(x)\,h(x) \tag{1}$$

in a region R which is a connected domain with smooth boundaries ∂R in an n-dimensional Euclidean space R_n, where $n \geqq 2$. Furthermore in R $m(x)$ is to be continuous and have a positive lower bound m, and ∂R is to lie entirely in the bounded part of R_n. Yosida's theorem then states that if $h(x)$ satisfies the internal boundary condition

$$\partial h/\partial n=0 \text{ on } \partial R, \tag{2}$$

and the order relation at infinity

$$h(x)=O(\exp(ar)), \text{ where } a\sqrt{2}<\sqrt{m}, \tag{3}$$

then $h(x)$ must vanish identically. Here I use r (in place of Yosida's $|x|$) to denote the distance from the origin of coordinates.

The aim of this note is to show the condition (3) may, by a slight modification of Yosida's argument, be replaced by what seems to be a best possible result in this direction. Consider namely the special case in which $n=3$, $m(x)=k^2$ where k is a positive constant, and in which ∂R is a sphere, centre the origin. We then have the spherically symmetric solutions

$$k(x)=r^{-1}\exp(\pm kr),$$

of which a non-trivial linear combination may be formed so as to satisfy (2). The mildest condition of the type of (3) which will exclude such solutions is

$$h(x)=o(r^{-1}\exp(kr)),$$

which is of course weaker than (3).

This example suggests that in the general case the condition (3) may be replaced by

$$h(x)=o(r^{-(1/2)(n-1)}\exp(r\sqrt{m})), \qquad (4)$$

and it is this that I prove in this paper.

2. It will be sufficient to consider the case in which R extends to infinity. As in Yosida's argument, let K_r denote a sphere, centre the origin, of radius r so large that K_r contains ∂R entirely. Let D_r denote the region between K_r and ∂R, and let ∂K_r denote the boundary of K_r. Let further $\partial h/\partial r$ denote the radial derivative of $h(x)$, in the sense of r increasing. Green's integral theorem then gives (here dv denotes the volume element, dS the surface element)

$$\int_{D_r}(h\Delta h+|\operatorname{grad} h|^2)dv=\int_{\partial K_r} h\, \partial h/\partial r\, dS.$$

We have here

$$h\Delta h=h^2m(x)\geqq h^2m,$$

and also

$$|\operatorname{grad} h|^2\geqq(\partial h/\partial r)^2,$$

so that

$$\int_{D_r}(mh^2+(\partial h/\partial r)^2dv\leqq\int_{\partial K_r} h\,\partial h/\partial r\, dS. \qquad (5)$$

Furthermore

$$h\,\partial h/\partial r\leqq\tfrac{1}{2}(h^2\sqrt{m}+(\partial h/\partial r)^2/\sqrt{m}),$$

and hence

$$\int_{D_r}(mh^2+(\partial h/\partial r)^2)dv\leqq\frac{1}{2\sqrt{m}}\int_{\partial K_r}(mh^2+(\partial h/\partial r)^2)dS. \qquad (6)$$

If then we define

$$J(r)=\int_{D_r}(mh^2+(\partial h/\partial r)^2)dv,$$

the result (6) states that

$$J(r)\leqq\frac{1}{2\sqrt{m}}J'(r).$$

It follows that the function

$$J(r)e^{-2r\sqrt{m}}$$

is a non-decreasing function of r, and hence, if $h(x)$ does not vanish identically, there will be a positive constant A such that

$$J(r) \geqq A e^{2r\sqrt{m}}$$

for sufficiently large r.

It now follows from (5) that

$$\int_{\partial K_r} h \, \partial h / \partial r \, dS \geqq A e^{2r\sqrt{m}}. \tag{7}$$

Following Yosida we define also

$$F(r) = \int_{D_r} h^2 dv,$$

so that

$$F'(r) = \int_{\partial K_r} h^2 dS,$$

$$F''(r) = \int_{\partial K_r} 2h \, \partial h / \partial r \, dS + \int_{\partial K_r} h^2 d(dS)/dr \geqq 2 \int h \, \partial h / \partial r \, dS,$$

whence, by (7),

$$F''(r) \geqq 2A e^{2r\sqrt{m}}.$$

Integrating twice over (r_0, r), where r_0 is some suitably large number, we derive

$$F(r) \geqq \frac{A}{2m} e^{2r\sqrt{m}} + Br + C,$$

where B, C are constants. Now if $h(x)$ does not vanish identically, A will be positive and the exponential term will predominate, so that for large r we shall have

$$F(r) \geqq A' e^{2r\sqrt{m}}, \tag{8}$$

where A' is some positive constant.

We show that (8) is incompatible with (4). Let r_1 be taken so large that ∂K_{r_1} encloses ∂R, and let $r > r_1$. Then

$$F(r) - F(r_1) = \int_{D_r - D_{r_1}} h^2 dv.$$

Splitting the latter integral up into elementary hyperspherical shells, and using (4) and the fact that

$$\int_{K_r} dS = O(r^{n-1})$$

we derive

$$F(r) \quad F(r_1) = \int_{r_1}^{r} o(\exp(r\sqrt{m}))dr = o(\exp(r\sqrt{m})),$$

which becomes contradictory with (8) as $r \to \infty$. This proves the result of the paper.