

**101. Probability-theoretic Investigations on Inheritance.**  
**XIV. Decision of Biovular Twins.<sup>1)</sup>**

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**1. Probability of deciding biovular twins.**

Both members of a pair of *monovular twins* possess always a mutually coincident type of any inherited character. Hence, they cannot be distinguished by means of an inherited character. However, types of an inherited character of both members of a pair of *biovular twins* are, on the contrary, regarded to show only a correlation based on a relationship that they have been originated from their common parents. Consequently, the distribution of an inherited character in biovular twins may be expected as the same as in ordinary brethren. In particular, it is to be postulated that, if both members of twins possess different types, then they must surely be biovular. In the present chapter, we shall derive the *probability of an event that a given pair of biovular twins can be decided to be surely biovular by means of an inherited character*, and make an application of the result.

The basic quantities required for solving the problem have already been obtained in a preceding chapter. In fact, we have derived in V the probabilities of brethren combinations belonging to the same family. In view of the postulate stated just above, all the pairs except those of coinciding types are decidable to be biovular. Hence, the desired probability of an event that the biovular twins given at random can be decided to be biovular is represented by

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1) Y. Komatu, Probability-theoretic investigations on inheritance. I. Distribution of genes; II. Cross-breeding phenomena; III. Further discussions on cross-breeding; IV. Mother-child combinations; V. Brethern combinations; VI. Rate of danger in random blood transfusion; VII. Non-paternity problems; VIII. Further discussions on non-paternity; IX. Non-paternity problems concerning mother-child combinations; X. Non-paternity concerning mother-child-child combinations; XI. Absolute non-paternity; XII. Problem of paternity; XIII. Estimation of genotypes. Proc. Japan Acad., **27** (1951); I. 371-377; II. 378-383, 384-387; III. 459-465, 466-471, 472-477, 478-483; IV. 587-592, 593-597, 598-603, 605-610, 611-614, 615-620; V. 639-633, 694-699; **28** (1952), VI. 54-58; VII. 102-104, 105-108, 109-111, 112-115, 116-120, 121-125; VIII. 162-164, 165-168, 169-171; IX. 207-212, 213-217, 218-223, 224-229; X. 249-253, 254-258, 259-264; XI. 311-316, 317-322; XII. 359-364, 365-369; XIII. 432-437, 438-443.

$$(1.1) \quad Y=1-\sum_{i=1}^m \sigma(ii, ii)-\sum'_{i,j} \sigma(ij, ij),$$

$\sigma$ 's denoting the probabilities of brethren combinations defined in (1.1) of V.

The second sum in the right-hand side of (1.1) represents the probability of an event that both members of biovular twins possess a coincident homozygotic type, while the third sum represents the one that they possess a coincident heterozygotic type.

Now, making use of the explicit expressions for the  $\sigma$ 's obtained in (1.5) and (1.9) of V, we get

$$(1.2) \quad \begin{aligned} \sum_{i=1}^m \sigma(ii, ii) &= \sum_{i=1}^m \frac{1}{4} p_i^2 (1+p_i)^2 \\ &= \frac{1}{4} (S_2 + 2S_3 + S_4), \end{aligned}$$

$$(1.3) \quad \begin{aligned} \sum'_{i,j} \sigma(ij, ij) &= \sum'_{i,j} \frac{1}{2} p_i p_j (1+p_i+p_j+2p_i p_j) \\ &= \frac{1}{4} (1 + S_2 - 2S_3 + 2S_2^2 - 2S_4). \end{aligned}$$

Thus, the desired probability (1.1) is expressed in the form

$$(1.4) \quad Y = \frac{1}{4} (3 - 2S_2 - 2S_2^2 + S_4).$$

By the way, it may be noticed that a half of the biovular twins can be distinguished previously based on the *sex discrepancy*. Hence, if we take this fact into account, the decision of the biovular twins as to be surely biovular will succeed with the probability

$$(1.5) \quad \frac{1}{2} (1 + Y) = 1 - \frac{1}{2} (1 - Y).$$

The same remark will also apply to the results which will be derived in the subsequent section.

## 2. Illustrative examples.

The above discussion concerns genotypes. We shall illustrate the corresponding problem in cases of several human blood types where recessive genes may be existent. The leading idea is quite the same as in the general discussion stated above. In fact, we have only to calculate the complementary probability of an event that both members of biovular twins possess a coincident type of an inherited character referred to. Rather explicitly stated, the sum of the terms lying on the principal diagonal in the table of brethren combinations belonging to the same family where the types of both members are arranged in the same order, has only to be subtracted from the unity, though the table on brethren combinations in concrete cases have been omitted in V; cf. the end of §1 of V.

Thus, in cases of *ABO blood type*, the probability of an event that both members of biovular twins can be decided really to be biovular is given in the form

$$\begin{aligned}
 Y_{ABO} &= 1 - \sigma(O, O) - \sigma(A, A) - \sigma(B, B) - \sigma(AB, AB) \\
 &= 1 - \frac{1}{4}r^2(1+r)^2 \\
 (2.1) \quad & - \frac{1}{4}p\{(p+r+p^2+3pr)(1+p+r) + (1+p)(1+r)r\} \\
 & - \frac{1}{4}q\{(q+r+q^2+3qr)(1+q+r) + (1+q)(1+r)r\} \\
 & - \frac{1}{2}pq(1+p+q+2pq) \\
 & = \frac{1}{2}r^2(1-r)(3+r) + \frac{1}{2}pq(4+6r-3pq).
 \end{aligned}$$

This result is quite identical with the one previously derived by Wiener and Leff<sup>2)</sup>. As shown above, the probability in question can be obtained quite briefly provided the probabilities of brethren combinations have been calculated. Wiener and Leff have derived their result more directly by means of a probability of each type of child produced from every mating.

They have derived also a result on *A<sub>1</sub>A<sub>2</sub>BO blood type* by a similar way. It can also be obtained briefly from the corresponding probabilities of brethren combinations. Though the explicit formulas for these probabilities have been omitted in V, a method of calculating them has been suggested there. Thus, we get the following result:

$$\begin{aligned}
 Y_{A_1A_2BO} &= 1 - \sigma(O, O) - \sigma(A_1, A_1) - \sigma(A_2, A_2) - \sigma(B, B) \\
 & - \sigma(A_1B, A_1B) - \sigma(A_2B, A_2B) \\
 & = 1 - \frac{1}{4}r^2(1+r)^2 \\
 & - \frac{1}{4}p_1\{(p_1+r+p_1^2+3p_1r)(1+p_1+r) + (1+p_1)(1+r)r \\
 & \quad + 2p_2(1+2p_1)(1+p_1+p_2+2r)\} \\
 & - \frac{1}{4}p_2\{(p_2+r+p_2^2+3p_2r)(1+p_2+r) + (1+p_2)(1+r)r\} \\
 & - \frac{1}{4}q\{(q+r+q^2+3qr)(1+q+r) + (1+q)(1+r)r\} \\
 & - \frac{1}{2}p_1q(1+p_1+q+2p_1q) - \frac{1}{2}p_2q(1+p_2+q+2p_2q),
 \end{aligned}$$

which after simplification becomes

$$(2.2) \quad Y_{A_1A_2BO} = Y_{ABO} + \frac{1}{2}p_1p_2((p_2+2r)(3+p_2+r) + q(2p_1+p_2+6q)),$$

coinciding with the result due to Wiener and Leff.

The probability on *MN blood type* is nothing but a special case ( $m=2$ ) of our general formula (1.4) which can be obtained rather briefly in a direct manner from the corresponding probabilities of brethren combinations. Thus, we get

2) A.S. Wiener and I.L. Leff, Chances of establishing the non-identity of biovular twins, with special reference to individuality tests of the blood. *Genetics* **25** (1940), 187-196. They pointed out there the errors contained in a previous note of the present author, What percent of biovular twins can one diagnose as biovular by blood type determination? (Japan.) *Hanzaigaku Zasshi* **13** (1939), 178-179.

$$\begin{aligned}
 Y_{MN} &= 1 - \sigma(M, M) - \sigma(N, N) - \sigma(MN, MN) \\
 (2.3) \quad &= 1 - \frac{1}{4}s^2(1+s)^2 - \frac{1}{4}t^2(1+t)^2 - st(1+st) \\
 &= \frac{1}{2}st(4-3st).
 \end{aligned}$$

This is a result once derived by the author<sup>3)</sup>. However, it had also already been contained in a paper of Wiener<sup>4)</sup>.

The probability on *Q blood type* becomes

$$\begin{aligned}
 Y_Q &= 1 - \sigma(Q, Q) - \sigma(q, q) \\
 (2.4) \quad &= 1 - \frac{1}{4}u(4+4uv+uv^2) - \frac{1}{4}v^2(1+v)^2 \\
 &= \frac{1}{2}uv^2(3+v),
 \end{aligned}$$

a result which has also been reported in a paper<sup>2)</sup> of Wiener and Leff. The probability on *Qq<sub>±</sub> blood type* can be derived in a similar manner, yielding the result

$$\begin{aligned}
 Y_{Qq_{\pm}} &= 1 - \sigma(Q, Q) - \sigma(q_-, q_-) - \sigma(q_+, q_+) \\
 (2.5) \quad &= 1 - \frac{1}{4}u(4+4uv+uv^2) \\
 &\quad - \frac{1}{4}v_1\{v_1(1+v_1)^2 + 2v_2(1+v)(1+2v_2)\} - \frac{1}{4}v_2^2(1+v_2)^2 \\
 &= Y_Q + \frac{1}{2}v_1v_2^2(2+v+v_2).
 \end{aligned}$$

By the way, we notice that the discontinuity of the sort explained in § 6 of VII does not appear between *ABO* and *MN*, *ABO* and *Q* as well as *A<sub>1</sub>A<sub>2</sub>BO* and *Qq<sub>±</sub>* types. The reason will be evidently seen from the way of their derivation.

### 3. Maximizing distributions.

We shall now determine the distribution which maximizes the respective probability *Y* obtained in the preceding sections.

First, in case of *MN* blood type, since the derivative

$$dY_{MN}/d(st) = 2 - 3st$$

remains positive everywhere in  $0 \leq st \leq 1/4$ , the maximizing distribution is given, as usual, by

$$(3.1) \quad s=t=1/2; \quad \bar{M}=\bar{N}=1/4, \quad \bar{MN}=1/2;$$

the maximum value being equal to

$$(3.2) \quad (Y_{MN})^{\max} = 13/32 = 0.40625.$$

In case of *ABO* blood type, we may regard the quantity *Y<sub>ABO</sub>* given in (2.1), in view of the relation  $p+q+r=1$ , as a function of two independent variables *r* and *pq*. Its partial derivative with respect to the latter variable satisfies then an inequality

$$\partial Y_{ABO} / \partial (pq) = 2 + 3r - 3pq > 0$$

for any admissible values of the variables, since

$$(3.3) \quad 0 \leq pq = ((1-r)^2 - (p-q)^2) / 4 \leq 1/4.$$

3) Y. Komatu, loc. cit.<sup>2)</sup>

4) A.S. Wiener, Heredity of agglutinogens *M* and *N* of Landsteiner and Levine. IV. Additional theoretico-statistical considerations. Human Biol. **7** (1935), 222-229.

Hence, the maximum will be attained at a distribution of the form

$$(3.4) \quad p=q=(1-r)/2, \quad 0 \leq r \leq 1.$$

Consequently, in order to maximize  $Y_{ABO}$ , we may observe it as a function of a single variable  $r$  alone which is expressed in the form

$$[Y_{ABO}]^{p=q=(1-r)/2} = \frac{1}{3^{1/2}}(1-r)(13+17r+15r^2+19r^3).$$

The cubic equation

$$(3.5) \quad 0 = -8d[Y_{ABO}]^{p=q=(1-r)/2}/dr = 19r^3 - 3r^2 + r - 1$$

yields, together with (3.4), the maximizing distribution

$$(3.6) \quad \begin{aligned} r &= 0.3816, & p &= q = 0.3092; \\ \bar{O} &= 0.1456, & \bar{A} &= \bar{B} = 0.3316, & \bar{AB} &= 0.1912; \end{aligned}$$

the maximum value being

$$(3.7) \quad (Y_{ABO})^{\max} = 0.4392.$$

In case of  $Q$  blood type, the probability may be regarded as a function of  $v=1-u$  alone. The equation

$$0 = dY_Q/dv = v(3-3v-2v^2)$$

yields the maximizing distribution

$$(3.8) \quad \begin{aligned} u &= (7-\sqrt{33})/4 = 0.3138, & v &= (\sqrt{33}-3)/4 = 0.6862; \\ \bar{Q} &= (3\sqrt{33}-13)/8 = 0.529, & \bar{q} &= (21-3\sqrt{33})/8 = 0.4718; \end{aligned}$$

the maximum value being

$$(3.9) \quad (Y_Q)^{\max} = 3(69-11\sqrt{33})/64 = 0.2733.$$

The discussion on  $A_1A_2BO$  and  $Qq_{\pm}$  blood types will be left to the reader.

In general case (1.4), the symmetric distribution

$$(3.10) \quad p_i = 1/m \quad (i=1, \dots, m)$$

is surely a stationary and perhaps the actually maximizing one for which it becomes

$$(3.11) \quad (Y)^{\text{stat}} = \frac{1}{4} \left( 1 - \frac{1}{m} \right) \left( 3 + \frac{1}{m} - \frac{1}{m^2} \right).$$

The last quantity increases steadily with  $m$  and tends to the limit  $3/4$  as  $m \rightarrow \infty$ .

#### 4. An application.

We have derived the probability  $Y$  of an event that biovular twins can be decided surely to be biovular based on an inherited character. The result can be applied to a problem which will be of some interest.

Let now the ratio of frequencies of monovular and biovular twins in a population be denoted by

$$(4.1) \quad \lambda_1 : \lambda_2 \quad (\lambda_1 + \lambda_2 = 1).$$

It will be estimated in a following manner.

In fact, let the pairs of twins, monovular as well as biovular, chosen at random in the population be examined with respect to an inherited character. Then, the pairs with different types are expected probabilistically to appear in the proportion

$$(4.2) \quad \eta \equiv \lambda_2 Y,$$

$Y$  being the probability defined in § 1, i.e., the proportion of pairs with different types among biovular twins. For (4.2), together with  $\lambda_1 + \lambda_2 = 1$ , we thus get

$$(4.3) \quad \lambda_1 : \lambda_2 = \left(1 - \frac{\eta}{Y}\right) : \frac{\eta}{Y}.$$

Since  $\eta$  is a quantity practically observable and  $Y$  is a quantity computable by means of a distribution of the character in the population. Hence, *the relation (4.3) will serve to estimate the ratio of frequencies of mono- and biovular twins.*