119. Probability-theoretic Investigations on Inheritance. XVI_1. Further Discussions on Interchange of Infants.[1)]

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1. Preliminaries.

A problem discussed in the preceding chapter has concerned a triple consisting of parents and an apparent infant as a unit of observation. On the other hand, we may consider an analogous problem if the type of one of parents in a triple, a father say, is unknown. The doubt on interchange of infants being very important for a family concerned, if such an affair happens, the data will be collected in detail as possible, and one of the most strong reference will be the father's inherited type. The previous discussions have just concerned such a circumstance. However, if a father's type cannot be known, for instance, on account of his death or disappearance, or when a rapid judgement must be brought, then there will arise a problem of detecting the interchange *without taking the father's type into account*; namely, a unit of observation is now a pair of a mother and an apparent infant. In the present chapter such a problem will be treated.

We first introduce a quantity analogous to (4.1) of XV. Let us designate by

$$(1.1)\qquad \psi(-ij,+hk) \qquad (i,j,h,k=1,\dots,m;\ (ij)\neq(hk))$$

the probability of an event that a mother unable to produce A_{ij} *appears and her child is* A_{hk}. Notation analogous to (4.2) of XV will also be availed. Though in order to detemine an explicit

1) Y. Komatu, Probability-theoretic investigations on inheritance. I. Distribution of genes; II. Cross-breeding phenomena; III. Further discussions on cross-breeding; IV. Mother-child combinations; V. Brethern ocmbinations; VI. Rate of danger in random blood transfusion; VII. Non-paternity problems; VIII. Further discussions on non-paternity; IX. Non-paternity problems concerning mother-children combinations; X. Non-paternity concerning mother-child-child combinations; XI. Absolute non-paternity; XII. Problem of paternity; XIII. Estimation of genotypes. XIV. Decision of biovular twins; XV. Detection of interchange of infants. Proc. Japan Acad., **27** (1951), I. 371-377; II. 378-383, 384-387; III. 459-465, 466-471, 472-477, 478-483; IV. 587-592, 593-597, 598-603, 605-610, 611-614, 615-620; V. 689-693, 694-699; **28** (1952), VI. 54-58; VII. 102-104, 105-108, 109-111, 112-115, 116-120, 121-125; VIII. 162-164, 165-168, 169-171; IX. 207-212, 213-217, 218-223; 224-229; X. 249-253, 254-258, 259-264; XI. 311-316, 317-322; XII. 359-364, 365-369, XIII. 432-437, 438-443; XIV. 444-449; XV. 517-520, 521-526, 527-532, 533-537.

expression for (4.1), we have made use of a table listed in § 3 of I, it must now be replaced by a table on mother-child combination listed in § 1 of IV.

First, for $\psi(-ii, +ih)(h \neq i)$, the mothers to be considered are

$$(1.2) \qquad A_{hi}, \qquad A_{hj} \qquad (j \neq i, h).$$

These types accompanied by a child A_{ih} appear with respective probabilities

$$(1.3) \qquad p_i p_h^2, \qquad p_i p_h p_j.$$

The sum of all the probabilities contained in (1.3) yields

$$(1.4) \qquad \psi(-ii, +ih) = p_i p_h^2 + \sum_{j \neq i, h} p_i p_h p_j = p_i p_h (1 - p_i).$$

A mother with a homozygotic child A_{ii} possesses necessarily a gene A_i and hence can produce a child A_{ih} also. Hence,

$$(1.5) \qquad \psi(-ih, +ii) = 0.$$

Next, for $\psi(-ii, +hh)$ $(h \neq i)$, mothers to be considered are again (1.2), but in this case, the probabilities (1.3) are to be replaced by

$$(1.6) \qquad p_h^3, \qquad p_h^2 p_j.$$

Thus, we get

$$(1.7) \qquad \psi(-ii, +hh) = p_h^3 + \sum_{j \neq i, h} p_h^2 p_j = p_h^2 (1 - p_i).$$

For $\psi(-ii, +hk)$ $(h, k \neq i; h \neq k)$, mothers to be considered are

$$(1.8) \qquad A_{hh}, \quad A_{kk}, \quad A_{hk}, \quad A_{hj}, \quad A_{kj} \qquad (j \neq i, h, k)$$

with corresponding probabilities of producing A_{hk}:

$$(1.9) \qquad p_h^2 p_k, \quad p_h p_k^2, \quad p_h p_k (p_h + p_k), \quad p_h p_k p_j, \quad p_h p_k p_j.$$

Thus, we get

$$(1.10) \qquad \begin{aligned} \psi(-ii, +hk) &= p_h^2 p_k + p_h p_k^2 + p_h p_k (p_h + p_k) + 2 \sum_{j \neq i, h, k} p_h p_k p_j \\ &= 2 p_h p_k (1 - p_i). \end{aligned}$$

For $\psi(-hk, +ii)$ $(h, k \neq i; h \neq k)$, mothers to be considered are

$$(1.11) \qquad A_{ii}, \qquad A_{ij} \qquad (j \neq i, h, k)$$

with corresponding probabilities of producing A_{ii}:

$$(1.12) \qquad p_i^3, \qquad p_i^2 p_j.$$

Thus, we get

$$(1.13) \qquad \psi(-hk, +ii) = p_i^3 + \sum_{j \neq i, h, k} p_i^2 p_j = p_i^2 (1 - p_h - p_k).$$

For $\psi(-ih, +ik)$ $(h, k \neq i; h \neq k)$, mothers to be considered are

$$(1.14)\qquad A_{kk},\qquad A_{kj}\qquad (j\neq i, h, k)$$

with corresponding probabilities of producing A_{ik}:

$$(1.15)\qquad p_i p_k^2,\qquad p_i p_k p_j.$$

Thus, we get

$$(1.16)\qquad \psi(-ih, +ik)=p_i p_k^2+\sum_{j\neq i, h, k} p_i p_k p_j = p_i p_k(1-p_i-p_h).$$

Last, for $\psi(-ij, +hk)$ $(i\neq j; h, k\neq i, j; h\neq k)$, mothers to be considered are

$$(1.17)\qquad A_{hh},\quad A_{kk},\quad A_{hk},\quad A_{hl},\quad A_{kl}\qquad (l\neq i, j, h, k)$$

with corresponding probabilities of producing A_{hk}:

$$(1.18)\qquad p_h^2 p_k,\quad p_h p_k^2,\quad p_h p_k(p_h+p_k),\quad p_h p_k p_l,\quad p_h p_k p_l.$$

Thus, we get

$$(1.19)\qquad \begin{aligned}\psi(-ij, +hk)&=p_h^2 p_k+p_h p_k^2+p_h p_k(p_h+p_k)+2\sum_{l\neq i, j, h, k} p_h p_k p_l\\ &=2p_h p_k(1-p_i-p_j).\end{aligned}$$

All the possible cases have thus been essentially worked out.

2. Main results.

The procedure of deriving main results is quite analogous to that availed in § 5 of XV. The only modification is that the table concerning the probabilities of mating-child combinations has to be replaced by the one concerning those of mother-child combinations.

Corresponding to (5.2) of XV, we denote by

$$(2.1)\qquad F_0(ij),\qquad \Psi(ij)$$

the probability of an event that a pair consisting of a mother A_{ij} and an apparent child is presented and the detection of interchange of infants is *possible against another pair within the first pair* or *only by taking the second pair into account*, respectively. The sum

$$(2.2)\qquad F(ij)=F_0(ij)+\Psi(ij)$$

represents the *probability of all cases of possible detection with a mother A_{ij} of the first pair.*

The former quantity in (2.1) can be determined in a very simple manner. In fact, for a homozygotic mother A_{ii}, the mother-child combinations with vanishing probability are those accompanied by the children $A_{hk}(h, k\neq i)$. Hence, we get

$$(2.3)\qquad F_0(ii)=\overline{A}_{ii}\sum_{h, k\neq i}\overline{A}_{hk}=p_i^2(1-p_i)^2.$$

For a heterozygotic mother $A_{ij}(i\neq j)$, such combinations are those accompanied by the children $A_{hk}(h, k\neq i, j)$. Hence, we get

$$(2.4)\qquad F_0(ij)=\overline{A}_{ij}\sum_{h, k\neq i, j}\overline{A}_{hk}=2p_i p_j(1-p_i-p_j)^2\qquad (i\neq j).$$

It would be noticed that (2.3) and (2.4) are identical with (1.2) and (1.3) of XI respectively, a fact which is evident from the definition.

The latter quantity in (2.1) can be determined by means of the preparations done in the preceding section. The procedure is quite analogous used for the latter quantity in (5.2) of XV. Thus, we obtain

$$\Psi(ii)=\bar{A}_{ii}\{p_i\psi(-ii,\ +\sum_{h\neq i} ih)+\sum_{h\neq i} p_h\psi(-ih,\ +ii+\sum_{k\neq i,h} ik)\}$$
$$=p_i^3(1-2S_2+S_3-2(1-S_2)p_i+3p_i^2-3p_i^3), \tag{2.5}$$

$$\Psi(ij)=\bar{A}_{ij}\{\tfrac{1}{2}p_i\psi(-ii,\ +jj+ij+\sum_{h\neq i,j}(ih+jh))+\tfrac{1}{2}p_j(-jj,\ +ii+ij$$
$$+\sum_{h\neq i,j}(ih+jh))+\tfrac{1}{2}(p_i+p_j)\psi(-ij,\ +ii+jj+\sum_{h\neq i,j}(ih+jh))$$
$$+\sum_{h\neq i,j}\tfrac{1}{2}p_h\psi(-ih,\ +ii+jj+ij+\sum_{k\neq i,j,h} ik+\sum_{k\neq i,j} jk) \tag{2.6}$$
$$+\sum_{h\neq i,j}\tfrac{1}{2}p_h\psi(-jh,\ +ii+jj+ij+\sum_{k\neq i,j} ik+\sum_{k\neq i,j,h} jk)\}$$
$$=p_ip_j((3-4S_2+S_3)(p_i+p_j)-(4-3S_2)(p_i^2+p_j^2)$$
$$-4(2-S_2)p_ip_j+4(p_i^3+p_j^3)+7p_ip_j(p_i+p_j)$$
$$-3(p_i^4+p_j^4)-3p_ip_j(p_i^2+p_j^2)-4p_i^2p_j^2) \qquad (i\neq j).$$

The sums of (2.3) and (2.5), and of (2.4) and (2.6) become

$$F(ii)=p_i^2(1-(1+2S_2-S_3)p_i-(1-2S_2)p_i^2+3p_i^3-3p_i^4), \tag{2.7}$$

$$F(ij)=p_ip_j(2-(1+4S_2-S_3)(p_i+p_j)-(2-3S_2)(p_i^2+p_j^2)$$
$$-4(1-S_2)p_ip_j+4(p_i^3+p_j^3)+7p_ip_j(p_i+p_j) \tag{2.8}$$
$$-3(p_i^4+p_j^4)-3p_ip_j(p_i^2+p_j^2)-4p_i^2p_j^2) \qquad (i\neq j).$$

Further summing up the probabilities (2.3), (2.4); (2.5) and (2.6) over respective possible suffices, we obtain

$$\sum_{i=1}^{m} F_0(ii)=S_2-2S_3+S_4, \tag{2.9}$$

$$\sum_{i,j}{}' F_0(ij)=1-5S_2+6S_3+2S_2^2-4S_4; \tag{2.10}$$

$$\sum_{i=1}^{m} \Psi(ii)=S_3-2S_4-2S_2S_3+3S_5+S_3^2+2S_2S_4-3S_6, \tag{2.11}$$

$$\sum_{i,j}{}' \Psi(ij)=3S_2-7S_3-8S_2^2+12S_4$$
$$+15S_2S_3-14S_5+2S_2^3-3S_3^2-8S_2S_4+8S_6. \tag{2.12}$$

Further summations yield

$$\sum_{i=1}^{m} F(ii)=S_2-S_3-S_4-2S_2S_3+3S_5+S_3^2+2S_2S_4-3S_6, \tag{2.13}$$

$$\sum_{i,j}{}' F(ij)=1-2S_2-S_3-6S_2^2+8S_4+15S_2S_3-14S_5$$
$$+2S_2^3-3S_3^2-8S_2S_4+8S_6; \tag{2.14}$$

$$F_0\equiv\sum_{i\leqq j} F_0(ij)=1-4S_2+4S_3+2S_2^2-3S_4, \tag{2.15}$$

$$\Psi\equiv\sum_{i\leqq j}\Psi(ij)=3S_2-6S_3-8S_2^2+10S_4+13S_2S_3-11S_5$$
$$+2S_2^3-2S_3^2-6S_2S_4+5S_6. \tag{2.16}$$

The sum of (2.13) and (2.14) or of (2.15) and (2.16) yields the *whole probability of detecting the interchange*:

$$F=F_0+\Psi$$
$$=1-S_2-2S_3-6S_2^2+7S_4+13S_2S_3 \tag{2.17}$$
$$-11S_5+2S_2^3-2S_3^2-6S_2S_4+5S_6.$$

—To be continued—