124. On a Theorem of R. Pallu de la Barrier.

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1. Let E be a compact Hausdorff space in which every open set has an open closure and μ be a regular positive measure on E such that every set of the first category has null measure. We call μ a *normal* positive measure and denote its support by $S(\mu)$, where a support of a measure means the complement of the join of open sets which have null measure. Furthermore, when E has sufficiently many normal positive measures such that the join of supports of these measures is dense in E, we call E a *hyperstonean space*. For detailed explanations for these definitions, we may refer Dixmier [1].

Let $\mathfrak{A}$ be a commutative W^*-algebra on a Hilbert space H. In the following we assume always $\mathfrak{A}$ contains an identity I. It is easy to see that the spectrum of $\mathfrak{A}$ (i.e. the space of maximal ideals of $\mathfrak{A}$ with Stone's topology) is a hyperstonean space in the above sense, and conversely a hyperstonean space can be characterized as such a spectrum (Dixmier [1]).

Furthermore, it has been introduced by J. Dixmier without proof that the following theorem has been demonstrated by R. Pallu de la Barrier. As it seems to be useful, we give here two kinds of its proof and its applications.

2. Theorem [R. Pallu de la Barrier].

Let E be a hyperstonean space and $\mathfrak{A}$ be a commutative W^*-algebra on H whose spectrum coincides with E. Then, for each normal complex measure τ, there exist two elements x, y of H such that

$$\int_E A(\gamma)d\tau = \langle Ax, y\rangle \tag{1}$$

for $A\in\mathfrak{A}$, where $A(\gamma)$ denotes the representative function on E of $A\in\mathfrak{A}$ and $\langle\ ,\ \rangle$ the inner product in H.

Lemma 1. For every hyperstonean space E_0, there exist a maximal abelian W^*-algebra $\mathfrak{A}_0$ on a Hilbert space H_0 whose spectrum coincides with E_0. For these H_0 and $\mathfrak{A}_0$, the theorem is true.

Proof. The existence of such H_0 and $\mathfrak{A}_0$ are already given by Dixmier [1]. From this, for each $x_i\in H_0$, we can define a normal positive measure μ_i on E_0 such as

$$\int_{E_0} A(\gamma) d\mu_i = \langle Ax_i, x_i \rangle \qquad \text{for } A \in \mathfrak{A}.$$

We take out so many such measures $\{\mu_i, (i \in I)\}$ on E satisfying the following two conditions:

(a) Supports of $\{\mu_i\}$ are disjoint each other,

(b) $R_0 = \bigcup \{S(\mu_i) ; i \in I\}$ is dense in E.

Then, denote by (R_0, μ) the direct sum of measure spaces $\{(S(\mu_i), \mu_i), i \in I\}$. As (R_0, μ) is a localizable space (Segal [6]), $L^\infty(R_0, \mu)$ considered as the mutiplication algebra on $L^2(R_0, \mu)$ is an maximal abelian W^*-algebra and its spectrum coincides with E_0 (Segal [6] and Dixmier [1]). So every maximal abelian W^*-algebra whose spectrum is homeomorphic to E_0 is unitary equivalent to $L^\infty(R_0, \mu)$.

By Dixmier [1], the total of normal measures on E_0 is isomorphic to $L^1(R_0, \mu)$ as a Banach space, so, for each normal measure τ on E_0, there exists a $f_\tau(\gamma) \in L^1(R_0, \mu)$ such as

$$\tau(A) = \int_{R_0} A'(\gamma) f_\tau(\gamma) d\mu$$

for $A \in \mathfrak{A}_0$, where $A'(\gamma)$ is the following function: $A'(\gamma) = A(\gamma)$ on R_0.

Define $\varphi(\gamma)$, $\psi(\gamma)$ as follows

$$\varphi(\gamma) = \begin{cases} \dfrac{f_\tau(\gamma)}{|f_\tau(\gamma)|^{1/2}} & \text{if } |f_\tau(\gamma)| \neq 0, \infty \\ 0 & \text{if } |f_\tau(\gamma)| = 0, \infty \end{cases} \qquad \psi(\gamma) = \begin{cases} |f_\tau(\gamma)|^{1/2} & \text{if } |f_\tau(\gamma)| \neq 0, \infty \\ 0 & \text{if } |f_\tau(\gamma)| = 0, \infty \end{cases}$$

then, $\varphi(\gamma) \in L^2(R_0, \mu)$, $\psi(\gamma) \in L^2(R_0, \mu)$ and

$$\tau(A) = \int_{R_0} A'(\gamma) \varphi(\gamma) \overline{\psi(\gamma)} d\mu = \langle Ax, y \rangle$$

where $x = U\varphi$, $y = U\psi$, U is an unitary transformation which defines the unitary equivalence between $L^\infty(R_0, \mu)$ and $\mathfrak{A}_0$. q.e.d.

Lemma 2. Let $\mathfrak{A}$ be a weakly closed subalgebra of a commutative W^*-algebra $\mathfrak{A}_0$ on H and E, E_0 be spectrums of $\mathfrak{A}$, $\mathfrak{A}_0$ respectively. Then E can be considered as the continuous image of E_0 by a mapping θ. Each $x_i \in H$, as in lemma 1, defines normal measures ν_i, μ_i on E, E_0 respectively and the relation between their supports is given by $\theta^{-1} \cdot S(\nu_i) \supset S(\mu_i)$.

Proof. The existence of the mapping θ is well known.

Let P be the projection in $\mathfrak{A}$ represented by the characteristic function of the open and closed set $E - S(\nu_i)$ in E, then the same projection P, considered as an openator in $\mathfrak{A}_0$, is represented in $C(E_0)$ as the characteristic function of $E_0 - \theta^{-1} \cdot S(\nu_i)$. So

$$\mu_i(E_0 - \theta^{-1} \cdot S(\nu_i)) = \langle Px_i, x_i \rangle = \nu_i(E - S(\nu_i)) = 0 .$$

Therefore, $\theta^{-1}\cdot S(\nu_i)\supset S(\mu_i)$. q.e.d.

In the following, for a function $f(\omega)$ on E, $f^*(\gamma)$ denotes a function on E_0 defined by $f^*(\gamma)=f(\omega)$, where $\gamma=\theta^{-1}\cdot w$.

Lemma 3. *For* $f(\omega)\in L^1(S(\nu_i), \nu_i)$,

$$\int_{S(\nu_i)} f(\omega)d\nu_i=\int_{S(\mu_i)} f^*(\gamma)d\mu_i \ .$$

Proof. Let F_α be an open-and-closed set in $S(\nu_i)$, then F_α and $\theta^{-1}\cdot F_\alpha$ denote a same projection P_α in $\mathfrak{A}$. Let $\{F_\alpha\}$ be the total family of open-and-closed sets in $S(\nu_i)$, then $\{F_\alpha\}$, $\{\theta^{-1}\cdot F_\alpha\}$ and $\{P_\alpha\}$ are isomorphic each other as complete Boolean algebras. Furthermore, let $P\in\mathfrak{A}_0$ be the projection represented by the characteristic function on $S(\mu_i)$, then the correspondence $P_\alpha\to P\cdot P_\alpha$ gives an isomorphism between complete Boolean algebras $\{P_\alpha\}$ and $\{P\cdot P_\alpha\}$. For if $PP_1-PP_2=0$, $P_1, P_2\in\{P_\alpha\}$ then

$$0=<P(P_1+P_2-2P_1P_2)x_i, x_i>=<(P_1+P_2-2P_1P_2)x_i, x_i>$$
$$=<(P_1-P_1P_2)x_i, x_i>+<(P_2-P_1P_2)x_i, x_i> \ .$$

So $P_1=P_1P_2=P_2$. Moreover, as $<PP_\alpha x_i, x_i>=<P_\alpha x_i, x_i>$,

$$\nu_i(F_\alpha)=\mu_i(\theta^{-1}\cdot F_\alpha\cap S(\mu_i)) \ .$$

So the lemma is clear.

First proof of theorem. Let $\mathfrak{A}_0$ be a maximal abelian W^*-algebra which contains $\mathfrak{A}$. We take out such $x_i\in H(i\in I)$ that the corresponding measures $\{\nu_i\}$ in E satisfy (a), (b) in Lemma 1. Then, by lemma 2, the supports of normal measures μ_i in E_0 are disjoint and $\bigcup\{S(\mu_i), i\in I\}$ is not always dense in E_0, so we adjoin $\{y_j\in H, j\in J\}$ to satisfy the density condition (b) in E_0. Put $R_0\equiv\bigcup_{i\in I}S(\mu_i)\bigcup\bigcup_{j\in J}S(\mu_j)$ and (R_0, μ) be the direct sum of measure spaces $(S(\mu_i), \mu_i)$, $(S(\mu_j), \mu_j)$ $(i\in I, j\in J)$. Then $\mathfrak{A}_0$ is unitary equivalent to the multiplication algebra $L^\infty(R_0, \mu)$ and the total family of normal measures on E_0 is isomorphic to $L^1(R_0, \mu)$.

Now, let (R, ν) be the direct sum of measure spaces $(S(\nu_i), \nu_i)$ $(i\in I)$ and τ be a general normal measure on E, then by lemma 3, there exists a $f(\omega)\in L^1(R, \nu)$ such that

$$\tau(A)\equiv\int_E A(\omega)d\tau=\int_{R'} A'(\omega)f(\omega)d\nu$$
$$=\int_{R_0} A^{*\prime}(\gamma)\cdot f^{*\prime\prime}(\gamma)d\mu \ ,$$

where denotes the contraction of a function to the support of the considering measure and

$$f^{*\prime\prime}=\begin{cases} f^{*\prime} & \text{on } \bigcup_{i\in I}S(\mu_i) \ , \\ 0 & \text{on } R_0-\bigcup_{i\in I}S(\mu_i) \ . \end{cases}$$

Clearly $f^{*\prime\prime}\in L^1(R_0, \mu)$, so it defines a normal measure E_0, then, by lemma 1, there exist $x, y\in H$ such as

$$\int_{R_0} A^{*\prime}(\gamma)f^{*\prime\prime}(\gamma)d\mu = <Ax, y>$$

for $A\in\mathfrak{A}$. So $\tau(A)=<Ax, y>$. q.e.d.

Corollary 1.1. For each normal positive measure τ on E, there exists an element x of H such as

$$\int_E A(\gamma)d\tau = <Ax, x>$$

for $A\in\mathfrak{A}$.

Proof. If τ is positive, by the Radon-Nikodym theorem, we can take $f(\omega)\geqq 0$ in the above proof. So, in this case, $\varphi(\gamma)=\psi(\gamma)$ in Lemma 1. From this the desired conclusion is easily obtained.

3. In this section, we give a simple alternative proof of the theorem from the stand point of functional analysis.

Second proof of theorem. Let $\mathfrak{A}_0$, E_0 be same as in the first proof. Then, from Lemma 1, each normal measure μ on E_0 is given by (1) as follows:

$$\int_{E_0} A(\omega)d\mu = <Ax, y> .$$

Let τ be the normal measure on E defined by

$$\int_E A(\gamma)d\tau = <Ax, y>$$

for $A\in\mathfrak{A}$, then the mapping π defined by $\pi(\mu)=\tau$ is a continuous linear operator from M (the total of normal measures on E_0) to N (the total of normal measures on E). If we can conclude $N=\pi(M)$, the proof is finished.

Let $C(E_0)$, $C(E_0)$ be the Banach space of all continuous complex valued functions on E, E_0 respectively, then by Dixmier [1], $C(E)$, $C(E_0)$ is the conjugate space of N and M respectively. The conjugate mapping π^* defined on $C(E)$ to $C(E_0)$ is nothing but the embedding of $\mathfrak{A}$ into $\mathfrak{A}_0$. This embedding is, of course, one-to-one and norm preserving, so it has a continuous inverse mapping. Then, by Banach's well known theorem, $N=\pi(M)$. q.e.d.

4. Applications.

1°. We get the following theorem. The terminology is due to Dye [2] and Nakamura—Takeda [4].

Theorem 2. Let $\mathfrak{A}$ be finite W^*-algebra on a separable Hilbert space H (or a σ-finite finite W^*-algebra). For a trace τ on $\mathfrak{A}$, following conditions are equivalent,

i) *τ is strongly sequentially continuous,*

ii) *τ is countably additive,*

iii) *the measure on the spectrum Ω of $\mathfrak{A}$ determined by τ is normal,*

iv) *there exists $x \in H$ such as $\tau(A)=\tau(A^{\eta})=\langle A^{\eta}x, x\rangle$ for $A \in \mathfrak{A}$.*

Proof. i)$\to$ii) Let $P=\sum_{i=1}^{\infty} P_i$, $p_i \in \mathfrak{A}$ be mutually disjoint projections, then $\{\sum_{i=1}^{n} p_i\}$ converges to P strongly. So, $\tau(\sum_{i=1}^{\infty} P_i)=\sum_{i=1}^{\infty}\tau(P_i)$.

ii)$\to$iii). If the measure determined by τ is not normal, there exist a set Γ of the first category with positive measure. From the conditions about $\mathfrak{A}$, there exist mutually disjoint open-and-closed sets in $\Omega-\Gamma$ at most countably. So there exists a sequence of central projections $\{P_i\}$ whose terms are mutually orthogonal, $I=\sum_{i=1}^{\infty} P_i$ and $\tau(I)\neq\sum_{i=1}^{\infty}\tau(P_i)$. This contradicts to the countable additivity of τ.

iii)$\to$iv). Clear by theorem 1 and Corollary 1.1.

iv)$\to$ i). Let $\{A_i\}$ be a sequence in $\mathfrak{A}$ which converges to A strongly. Then $\| A_i \|$ is bounded, so $\{A_i^{\eta}\}$ converges to A^{η} strongly. Since

$$\tau(A_i-A)=\tau(A_i^{\eta}-A^{\eta})=\langle(A_i^{\eta}-A^{\eta})x, x\rangle \leqq \|(A_i^{\eta}-A^{\eta})x\|\cdot\|x\| \to 0,$$

τ is strongly sequentially continuous.

2°. The essential part of Theorem 5 in Segal [5] is obtained without multiplicity theory.

Theorem 3 [*I. E. Segal*]. *For a commutative W^*-algebra $\mathfrak{A}$, there exists a maximal abelian W^*-algebra which is algebraically isomorphic and weakly bicontinuous to $\mathfrak{A}$.*

Proof. Let E be the spectrum of $\mathfrak{A}$, then $\mathfrak{A}$ is isomorphic to $C(E)$ as a C^*-algebra and, from the proof of theorem 1, $L^{\infty}(R, \nu)$ is a maximal abelian W^*-algebra on $L^2(R, \nu)$ whose spectrum coincides with E, then $L^{\infty}(R, \nu)$ is isomorphic to $C(E)$, also. We denote these isomorphism η, ζ respectively. Let $U(A_0; x_i, y_i\,(i=1, 2, \cdots, n); \varepsilon)$ be a weak neighborhood of $A_0 \in \mathfrak{A}$ and τ_i be a normal measure on E defined by $\tau_i(A^{\eta})=\langle Ax_i, y_i\rangle$. Because $C(E)$ is the conjugate space of the Banach space consisting of all normal measures on E,

$$V(A_0^{\eta}; \tau_i(i=1, 2, \cdots, n); \varepsilon)\equiv\{A^{\eta}: |\tau_i(A^{\eta}-A_0^{\eta})|<\varepsilon\, i=1, 2, \cdots, n\}$$

is a weak*-neighborhood of A_0^{η} in the Banach space $C(E)$. Clearly,

$$V(A_0^{\eta}; \tau_i; \varepsilon)=\{A^{\eta} : A \in U(A_0; x_i, y_i; \varepsilon)\}\,. \tag{2}$$

By theorem 1, for any weak*-neighborhood of $C(E)$, there exists a weak neighborhood of the operator algebra $\mathfrak{A}$ which satisfies (2). Therefore, η is bicontinuous with respect to the weak topology of the operator algebra $\mathfrak{A}$ and the weak*-topology of the

Banach space $C(E)$.

Alike, ζ is bicontinuous with respect to the weak topology of the operator algebra $L^\infty(R, \nu)$ and the weak*-topology of the Banach space $C(E)$, so $\zeta^{-1}\eta$ is an algebraic isomorphism and is bicontinuous with respect to weak topologies of each operator algebra.

Corollary 3.1. Let $\mathfrak{A}$ be a commutative AW^*-algebra [3] considered as an operator algebra on a Hilbert space H. If the measure ν defined by $\nu(A^\eta)=<Ax, y>$ is always normal, then $\mathfrak{A}$ is weakly closed as an operator algebra.

Proof. If the assumption is satisfied, the spectrum of a commutative AW^*-algebra $\mathfrak{A}$ is hyperstonean. Then by the same reasoning as above, we get a maximal abelian W^*-algebra $\mathfrak{M}$ which is isomorphic to $\mathfrak{A}$ and bicontinuous with respect to weak topologies. We denote this isomorphism by σ.

Let B be a weak limit of a directed set $\{A_\alpha\}$ in $\mathfrak{A}$, then $\{A_\alpha^\sigma\}$ converges to $A'\in\mathfrak{M}$ weakly and there exists such $A\in\mathfrak{A}$ as $A^\sigma=A'$. Clearly $B=A$, so the operator algebra $\mathfrak{A}$ is weakly closed.

Remark. J. Dixmier [1] discussed the existence of a non-hyperstonean Boolean space of a complete Boolean algebra. Let E be such a space, then $C(E)$ gives an example of an AW^*-algebra for which any representation as an operator algebra is not weakly closed.

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References.

1) J. Dixmier: Sur certains espaces considers par M. H. Stone. Summa Brasiliensis Math. **2** (1951).

2) H. A. Dye: The Radon-Nikodym theorem for finite rings of operators. Trans. Amer. Math. Soc. **72** (1952).

3) I. Kaplansky: Projections in Banach algebras. Ann. Math. **53** (1951).

4) M. Nakamura and Z. Takeda: The Radon-Nikodym theorem of traces for a certain operator algebras, to appear in Tōhoku Math. Jour.

5) I. E. Segal: Decomposition of operator algebras II. Multiplicity theory. Memoires Amer. Math. Soc. (1951).

6) I. E. Segal: Equivalence of measure spaces. Amer. Jour. Math. **73** (1951).