11. *Probability-theoretic Investigations on Inheritance. XVI$_4$. Further Discussions on Interchange of Infants*

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8. Comparison between several probabilities

We state here some inequalities supplementing those mentioned in § 3. From their respective definitions, we get immediately the inequalities

$$(8.1) \qquad \Psi(ij) \leqq \Phi_*(ij), \qquad \Psi_*(ij,\ hk) \leqq \Phi(ij,\ hk),$$

and consequently

$$(8.2) \qquad F(ij) \leqq \mathfrak{F}(ij), \qquad \mathfrak{G}(ij,\ hk) \leqq G(ij,\ hk).$$

We thus conclude the inequalities $\Psi \leqq \Phi_*$, $\Psi_* \leqq \Phi$ and hence

$$(8.3) \qquad F \leqq \mathfrak{F} \equiv \mathfrak{G} \leqq G,$$

which are also evident by definitions. These inequalities can also be verified directly from their final expressions. For instance, by rearranging the terms in $\mathfrak{F}-F$, we get

$$\begin{aligned}
\mathfrak{F}-F &= (S_2-S_3)+3(S_3-S_4)+3(S_2^2-S_4)-(S_4-S_5)-18(S_2S_3-S_5) \\
&\quad -(S_5-S_6)-9S_2(S_2^2-S_4)+17(S_2S_4-S_6)+8(S_3^2-S_6)+8S_3(S_2^2-S_4) \\
&\qquad +4(S_3S_4-S_7)-12(S_2S_5-S_7) \\
&= \sum_{i,j}{}' p_ip_j\{(p_i+p_j)+3(p_i^2+p_j^2)+6p_ip_j-(p_i^3+p_j^3)-18p_ip_j(p_i+p_j) \\
&\quad -(p_i^4+p_j^4)-18S_2p_ip_j+17p_ip_j(p_i^2+p_j^2)+16p_i^2p_j^2+16S_3p_ip_j \\
&\qquad +4p_i^2p_j^2(p_i+p_j)-12p_ip_j(p_i^3+p_j^3)\} \\
&= \sum_{i,j}{}' p_ip_j\{2(p_i^2+p_j^2)(p_i-p_j)^2+4p_ip_j(p_i+p_j)(p_i^2+p_j^2) \\
&\qquad +18p_ip_j((1-p_i-p_j)^2-(S_2-p_i^2-p_j^2)) \\
&\quad +16p_ip_j(S_3-p_i^3-p_j^3)+(p_i+p_j)(8(p_i^2+p_j^2)+p_ip_j)(1-p_i-p_j) \\
&\quad +6((p_i-p_j)^2+p_ip_j)(1-p_i-p_j)^2+(p_i+p_j)(1-p_i-p_j)^3\},
\end{aligned}$$

the last member remaining evidently always non-negative, since

$$(1-p_i-p_j)^2-(S_2-p_i^2-p_j^2)=\sum_{h,k\neq i,j}{}' 2p_hp_k \geqq 0.$$

For the difference $G-\mathfrak{G}$, we get similarly

$$\begin{aligned}
G-\mathfrak{G} &= 2S_2(S_2-S_3)+2(S_2S_3-S_5)+2S_2(S_3-S_4) \\
&\qquad +2S_2(S_2^2-S_4)-2(S_5-S_6) \\
&\quad -8(S_2S_4-S_6)-5(S_3^2-S_6)-16S_2(S_2S_3-S_5) \\
&\qquad +12(S_2S_5-S_7)+16(S_3S_4-S_7) \\
&\quad -4S_2^2(S_2^2-S_4)+12S_2(S_2S_4-S_6)-5(S_4^2-S_8)-4(S_2S_6-S_8)
\end{aligned}$$

$$=\sum_{i,j}{}' p_i p_j \{2S_2(p_i+p_j)+2p_ip_j(p_i+p_j)+2S_2(p_i^2+p_j^2) \\ +4S_2p_ip_j-2(p_i^4+p_j^4) \\ -8p_ip_j(p_i^2+p_j^2)-10p_i^2p_j^2-16S_2p_ip_j(p_i+p_j) \\ +12p_ip_j(p_i^3+p_j^3)+16p_i^2p_j^2(p_i+p_j) \\ -8S_2^2p_ip_j+12S_2p_ip_j(p_i^2+p_j^2)-10p_i^3p_j^3-4p_ip_j(p_i^4+p_j^4)\}$$

$$=\sum_{i,j}{}' p_ip_j\{2(p_i+p_j)(p_i^5+p_j^5)+4p_i^2p_j^2(p_i^2+p_j^2)\ +6p_i^3p_j^3 \\ +8S_2p_ip_j\sum_{h,k\neq i,j}{}' 2p_hp_k+2((p_i^3+p_j^3)+p_ip_j(p_i^2+p_j^2) \\ +2(p_i^5+p_j^5)+3p_ip_j(p_i^3+p_j^3)+9p_i^2p_j^2)(1-p_i-p_j) \\ +2(p_i^4+p_j^4)(1-p_i-p_j)^2+4p_ip_j(p_i+p_j)(1-p_i-p_j)^3 \\ +(((p_i+p_j)(2+p_i+p_j+4p_ip_j)+2(p_i^2+p_j^2)(1+p_i+p_j))(1-p_i-p_j) \\ +(p_i-p_j)^2(1+p_i+p_j+2(p_i+p_j)^2))(S_2-p_i^2-p_j^2)\},$$

which remains also evidently non-negative.

9. Illustration by concrete examples

The general result reduces to the one for *MN blood type* by putting merely $m=2$ and simplifying by means of the recurrence relation $S_\nu=S_{\nu-1}-stS_{\nu-2}$. However, for this particular case, the desired result will be obtained rather briefly by direct calculation. We thus get

$$(9.1)\quad \begin{aligned} &\Psi_*(M,M)=0,\ \Psi_*(M,N)=\Psi_*(N,M)=0, \\ &\qquad\Psi_*(M,MN)=\Psi_*(MN,M)=s^4t^3, \\ &\Psi_*(N,N)=0,\ \Psi_*(N,MN)=\Psi_*(MN,N)=s^3t^4, \\ &\qquad\Psi_*(MN,MN)=s^2t^2(1-2st); \end{aligned}$$

$$(9.2)\quad \Phi_*(M)=s^4t(2t+s^2),\ \Phi_*(N)=st^4(2s+t^2), \\ \Phi_*(MN)=st(1-3st+5s^2t^2);$$

$$(9.3)\quad \Psi_{*MN}=s^2t^2,\quad \Phi_{*MN}=2st(1-3st+3s^2t^2);$$

$$(9.4)\quad \mathfrak{G}_{MN}\equiv\mathfrak{F}_{MN}=2st(1-2st+3s^2t^2).$$

The inequalities specifying (8.3) are here quite obvious, namely we have

$$(9.5)\quad \mathfrak{F}_{MN}-F_{MN}=2st(1-2st)^2,\ G_{MN}-\mathfrak{G}_{MN}=2st((1-st)^2(1-3st)+2s^3t^3).$$

The case where recessive genes exist can be treated due to the same principle. In case of *ABO blood type*, we get in turn

$$(9.6)\quad \begin{aligned} &\Psi_*(O,O) && &&=0, \\ &\Psi_*(O,A) &&=\Psi_*(A,O) &&=pr^3\cdot pq(p+r), \\ &\Psi_*(O,B) &&=\Psi_*(B,O) &&=qr^3\cdot pq(q+r), \\ &\Psi_*(O,AB) &&=\Psi_*(AB,O)&&=0, \\ &\Psi_*(A,A) && &&=p^2r^2\cdot pq(p+r), \\ &\Psi_*(A,B) &&=\Psi_*(B,A) &&=pqr^2\cdot 2pq+pq(p+r)(q+r)\cdot r^2, \\ &\Psi_*(A,AB) &&=\Psi_*(AB,A)&&=p^2q(p+r)\cdot r^2(p+q), \\ &\Psi_*(B,B) && &&=q^2r^2\cdot pq(q+r), \\ &\Psi_*(B,AB) &&=\Psi_*(AB,B)&&=pq^2(q+r)\cdot r^2(p+q), \\ &\Psi_*(AB,AB) && &&=2p^2q^2\cdot r^2(p+q); \end{aligned}$$

$$
(9.7)\quad
\begin{aligned}
\Phi_*(O) &= r^3(4pqr^2+2p^2q(p+3r)+2pq^2(q+3r)+2p^2q^2)\\
&\quad+pr^2(r^4+2qr^2(q+2r)+q^2(q+2r)^2)\\
&\quad+qr^2(r^4+2pr^2(p+2r)+p^2(p+2r)^2),\\
\Phi_*(A) &= pr^2(4pqr^2+4p^2q(p+2r)+4pq^2(q+2r)+4p^2q^2)\\
&\quad+p(p^2+3pr+r^2)(r^4+2qr^2(q+2r)+q^2(q+2r)^2)\\
&\quad+pqr(r^4+2pr^2(p+2r)+p^2(p+2r)^2)\\
&\quad+pq(p+r)(r^4+2pr^2(p+2r)+2qr^2(q+2r)+4pqr^2\\
&\qquad\qquad+p^2(p+2r)^2+q^2(q+2r)^2),\\
\Phi_*(B) &= qr^2(4pqr^2+4p^2q(p+2r)+4pq^2(q+2r)+4p^2q^2)\\
&\quad+pqr(r^4+2qr^2(q+2r)+q^2(q+2r)^2)\\
&\quad+q(q^2+3qr+r^2)(r^4+2pr^2(p+2r)+p^2(p+2r)^2)\\
&\quad+pq(q+r)(r^4+2pr^2(p+2r)+2qr^2(q+2r)+4pqr^2\\
&\qquad\qquad+p^2(p+2r)^2+q^2(q+2r)^2),\\
\Phi_*(AB) &= pq(p+r)(2qr^2(q+r)+q^2(q+r)(q+3r))\\
&\quad+pq(q+r)(2pr^2(p+r)+p^2(p+r)(p+3r))\\
&\quad+pq(p+q)(2pr^2(p+r)+2qr^2(q+r)+4pqr^2\\
&\qquad\qquad+p^2(p+r)(p+3r)+q^2(q+r)(q+3r)).
\end{aligned}
$$

$$
(9.8)\quad
\begin{aligned}
\Psi_{*ABO} &= pqr^2(3-2r+r^2+pq(1-r));\\
\Phi_{*ABO} &= r^4(1-r^2)+pq(1+2r+5r^2+4r^3+5r^4)\\
&\qquad -p^2q^2(7+12r+4r^2+r^3)+2p^2q^2;
\end{aligned}
$$

$$
(9.9)\quad
\begin{aligned}
\mathfrak{G}_{ABO} \equiv \mathfrak{F}_{ABO} &= r^4(1-r^2)+pq(2+2r+9r^2+4r^3+5r^4)\\
&\qquad -p^2q^2(7+12r+4r^2+r^3)+2p^3q^3.
\end{aligned}
$$

The inequalities corresponding to (8.3) are valid here also:

$$
(9.10)\quad
\begin{aligned}
\mathfrak{F}_{ABO}-F_{ABO} &= r^4(1-r^2)+pq(2+2r+r^2+6r^3+3r^4)\\
&\qquad -p^2q^2(7+12r+4r^2+r^3)+2p^3q^3 \geqq 0,\\
G_{ABO}-\mathfrak{G}_{ABO} &= r^4(1-r^2)+pq(2+2r-r^2+4r^3-5r^4)\\
&\quad -p^2q^2(7+12r+6r^2+19r^3-2r^4)+2p^3q^3(1-4r-8r^2)-2p^4q^4 \geqq 0
\end{aligned}
$$

as readily seen in view of $pq \leqq (1-r)^2/4$.

In cases of *Q and $Qq_\pm$ blood types*, there are no mother-child combinations with vanishing probability. Hence, we get

$$(9.11)\qquad \Psi_{*Q}=\Psi_{*Qq\pm}=0.$$

But, by remembering the facts that the mating $q\times q$ cannot produce a child Q and further the mating $q_+\times q_+$ cannot produce a child q_-, we get

$$(9.12)\qquad \Phi_{*Q}=v^4\cdot u(1+v)=uv^4(1+v),$$

$$(9.13)\qquad \Phi_{*Qq\pm}=v^4\cdot u(1+v)+v_2^4\cdot v_1(v+v_2)=uv^4(1+v)+v_1v_2^4(v+v^2).$$

Thus, we obtain

$$(9.14)\qquad \mathfrak{G}_Q \equiv \mathfrak{F}_Q = uv^4(1+v),$$

$$(9.15)\qquad \mathfrak{G}_{Qq\pm} \equiv \mathfrak{F}_{Qq\pm} = uv^4(1+v)+v_1v_2^4(v+v_2);$$

these values are equal to a half of G_Q and of $G_{Qq\pm}$, respectively. Since $F_Q=F_{Qq\pm}=0$, we have

(9.16) $\mathfrak{F}_Q - F_Q = G_Q - \mathfrak{G}_Q = \frac{1}{2}G_Q, \quad \mathfrak{F}_{Qq\pm} - F_{Qq\pm} = G_{Qq\pm} - \mathfrak{G}_{Qq\pm} = \frac{1}{2}G_{Qq\pm}.$

The discontinuity between ABO and MN types appears here also. In fact,

(9.17) $$[\mathfrak{G}_{ABO}]^{r=0} - [\mathfrak{G}_{MN}]^{(s,t)=(p,q)} = -p^3q^3(3+4pq) \leqq 0.$$

But, there is no discontinuity between ABO and Q as well as $Qq_\pm$ and Q types.

10. Maximizing distributions

The distribution of genes maximizing the respective probability will be determined by means of a usual procedure.

Differentiating the probability (9.4) with respect to the variable st, we get

(10.1) $$d\mathfrak{G}_{MN}/d(st) = 2(1-4st+9s^2t^2) \qquad (0 \leqq st \leqq 1/4).$$

Hence, the maximizing distribution is given by

(10.2) $$s=t=1/2; \qquad \overline{M}=\overline{N}=1/4, \qquad \overline{MN}=1/2,$$

yielding the maximum value

(10.3) $$(\mathfrak{G}_{MN})^{\max} = 11/32 = 0.34375.$$

The probability (6.21) in general case attains its stationary value

(10.4) $$(\mathfrak{G})^{\text{stat}} = 1 - \frac{4}{m^2} - \frac{7}{m^3} + \frac{33}{m^4} - \frac{31}{m^5} + \frac{8}{m^6}$$

for the symmetric distribution

(10.5) $$p_i = 1/m \qquad (i=1,\ldots, m),$$

which will perhaps yield the actual maximum. The value (10.4) increases with m and tends to 1 as $m\rightarrow\infty$. In fact,

$$\frac{d}{d(1/m)}(\mathfrak{G})^{\text{stat}} = -\frac{1}{m}\left(1-\frac{2}{m}\right)\left(8+\frac{8}{m}+\frac{29}{m}\left(1-\frac{2}{m}\right)+\frac{39}{m^3}\right)-\frac{30}{m^5}.$$

In case of ABO type, the probability (9.9) may be regarded, in view of an identity $p+q+r=1$, as a function of two independent variables p and q. The system of equations $\partial\mathfrak{G}_{ABO}/\partial p = \partial\mathfrak{G}_{ABO}/\partial q = 0$ will then imply the maximizing distribution given by

(10.6) $$p=q=0.2172, \qquad r=0.5656;$$
$$\overline{O}=0.3199, \qquad \overline{A}=\overline{B}=0.2929, \qquad \overline{AB}=0.0943;$$

the value of p and q in (10.6) is a root of the equation of degree six:

(10.7) $$28x^6-30x^5+115x^4-88x^3+36x^2-14x+2=0.$$

The maximum value of the probability is equal to

(10.8) $$(\mathfrak{G}_{ABO})^{\max} = 0.3777.$$

In cases of Q and $Qq_\pm$ types, no further discussion is necessary. In fact, since the whole probabilities (9.14) and (9.15) are equal to a half of the corresponding G-probabilities in (7.12) and (7.16) of XV, the maximizing distributions are the same as in (9.8) and (9.10) of XV, namely

$$u=1-\sqrt{2/3}=0.1723, \qquad v=\sqrt{2/3}=0.8277,$$

(10.9) $$\overline{Q}=1/3=0.3333, \qquad \overline{q}=2/3=0.6667;$$

$$u=1-3\sqrt{2/23}=0.1154, \qquad v=3\sqrt{2/23}=0.8846,$$

(10.10) $$v_1=3\sqrt{2/23}-2\sqrt{3/23}=0.0480, \qquad v_2=2\sqrt{3/23}=0.8366,$$

$$\overline{Q}=5/23=0.2174, \quad \overline{q}_-=6/23=0.2609, \quad \overline{q}_+=12/23=0.5217.$$

The maximum values are equal to

(10.11) $$(\mathfrak{G}_Q)^{\max}=4/27=0.1481;$$

(10.12) $$(\mathfrak{G}_{Qq\pm})^{\max}=108/529=0.2042.$$

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