37. On the Jordan-Hölder-Schreier Theorem

By Tsuyoshi FUJIWARA and Kentaro MURATA
Department of Mathematics, Yamaguchi University
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In this note we shall formulate the Jordan-Hölder-Schreier Theorem for groups in any lattice. This formulation is the extension of the usual Jordan-Hölder-Schreier Theorem for modular lattices, and of the Jordan-Hölder Theorem for composition series of lower semi-modular lattices.

Let L be a lattice. In the following we denote the elements of L by small latters $a, b, x, y, m, n, \cdots$. By m/n we mean the closed interval $\{x;\ m\geqq x\geqq n,\ x\in L\}$, and by $m/$ the principal ideal generated by m in L.

Definition 1. An element $a\in m/n$ is called m/n-modular if and only if

1) $x, y\in m/n$, $x\geqq a$ implies $(x\frown y)\smile a=x\frown(y\smile a)$ and
2) $x, y\in m/n$, $x\geqq y$ implies $(x\frown a)\smile y=x\frown(a\smile y)$.

Remark. Putting $m/$-modular in place of m/n-modular in this definition, we can argue similarly in the following arguments.

Theorem 1. If $a, b\in m/n$ and a is m/n-modular, then the correspondences $x\to x\frown b$ and $y\to y\smile a$ are inverse isomorphisms between $a\smile b/a$ and $b/a\frown b$.

Proof. This theorem is immediate from the above definition.

Theorem 2. If a is m/n-modular and $b\in m/n$, then $a\frown b$ is b/n-modular.

Proof. (i) If $x, y\in b/n$ and $x\geqq a\frown b$ then

$$\begin{aligned}
&(x\frown y)\smile(a\frown b)\\
&=[(x\frown y)\smile a]\frown b\\
&=\{[(a\smile x)\frown b\frown y]\smile a\}\frown b \qquad \text{(applying Theorem 1)}\\
&=(a\smile x)\frown[(b\frown y)\smile a]\frown b\\
&\geqq x\frown(y\smile a)\frown b\\
&=x\frown[y\smile(a\frown b)]\\
&\geqq(x\frown y)\smile(a\frown b).
\end{aligned}$$

Hence we have

$$(x\frown y)\smile(a\frown b)=x\frown[y\smile(a\frown b)]$$

(ii) If $x, y\in b/n$ and $x\geqq y$, then we have

$$\begin{aligned}
&x\frown[(a\frown b)\smile y]\\
&=x\frown[b\frown(a\smile y)]\\
&=x\frown(a\smile y)\\
&=(x\frown a)\smile y\\
&=[x\frown(a\frown b)]\smile y.
\end{aligned}$$

This complete the proof.

Theorem 3. Let $n \leqq a \leqq b \leqq m$. If a is m/n-modular and b m/a-modular then b is m/n-modular.

Proof. (i) If $x, y \in m/n$ and $x \geqq b$, then we have

$$\begin{aligned} & x \frown (y \smile b) \\ &= [(a \smile y) \frown x \frown (y \smile b)] \smile b \qquad \text{(applying Theorem 1)} \\ &= [(a \smile y) \frown x] \smile b \\ &= [a \smile (y \frown x)] \smile b \\ &= (x \frown y) \smile b. \end{aligned}$$

(ii) If $x, y \in m/n$ and $x \geqq y$, then we have

$$\begin{aligned} & x \frown (b \smile y) \\ &= (x \smile a) \frown (b \smile y \smile a) \frown x \\ &= \{[(x \smile a) \frown b] \smile (y \smile a)\} \frown x \\ &= [a \smile (x \frown b) \smile (y \smile a)] \frown x \\ &= [a \smile (x \frown b) \smile y] \frown x \\ &= (x \frown b) \smile y. \qquad \text{(applying Theorem 1)} \end{aligned}$$

This complete the proof.

Theorem 4. Let a be m/n-modular and $b \in m/n$.

1) If $x \in a \smile b/a$ and $a \smile b/n$-modular then $x \frown b$ is b/n-modular.

2) If $y \in b/a \frown b$ and b/n-modular then $y \smile a$ is $a \smile b/n$-modular.

Proof. This theorem is immediate from the above three theorems.

Theorem 5. If a, b are m/n-modular, then $a \smile b$ is m/n-modular.

Proof. (i) If $x, y \in m/b$ and $x \geqq a \smile b$, then we have

$$\begin{aligned} & (x \frown y) \smile (a \smile b) \\ &= [x \frown (y \smile a)] \smile b \\ &= x \frown [y \smile (a \smile b) \ . \end{aligned}$$

(ii) If $x, y \in m/b$ and $x \geqq y$, then

$$\begin{aligned} & [x \frown (a \smile b)] \smile y \\ &\geqq (x \frown a) \smile y \\ &= x \frown (a \smile y) \\ &= x \frown [(a \smile b) \smile y] \\ &\geqq [x \frown (a \smile b)] \smile y. \end{aligned}$$

i.e. $[x \frown (a \smile b)] \smile y = x \frown [(a \smile b) \smile y]$.

Hence a$\smile b$ is m/b-modular. Using Theorem 3, we conclude the m/n-modularity of $a \smile b$.

Definition 2. Let $m = a_0 > a_1 > \cdots > a_r = n \geqq n_0$ be a chain such that a_i is a_{i-1}/n_0-modular $(i=1, \cdots, r)$. We call such a chain a m/n-modular chain on n_0.

Theorem 6. (Schreier's Theorem). Let

$$m = a_0 > a_1 > \cdots > a_r = n \geqq n_0$$

and

$$m = b_0 > b_1 > \cdots > b_s = n \geqq n_0$$

be any two finite m/n-modular chains on n_0, then these modular chains can be refined by interpolation of terms $a_{i,j} = a_{i+1} \smile (a_i \frown b_j)$ and $b_{i,j} =$

$b_{j+1}\cup(a_i\cap b_j)$ so that corresponding intervals $a_{i,j}/a_{i,j+1}$ and $b_{i,j}/b_{i+1,j}$ are projective and isomorphic.

Proof. (i) Proof of refinement:
a_{i+1} is a_i/n_0-modular. Hence by Theorem 1 we have

(1) $$a_{i,j}/a_{i+1}\cong a_i\cap b_j/a_{i+1}\cap b_j\,.$$

Moreover, using Theorem 2 we have that $a_{i+1}\cap b_j$ is $a_i\cap b_j/n_0$-modular. Similarly $a_i\cap b_{j+1}$ is $a_i\cap b_j/n_0$-modular. Hence applying Theorem 5, we have that:

(2) $$(a_{i+1}\cap b_j)\cup(a_i\cap b_{j+1}) \text{ is } a_i\cap b_j/n_0\text{-modular.}$$

Since

(3) $$a_{i+1}\cup(a_{i+1}\cap b_j)\cup(a_i\cap b_{j+1})=a_{i+1}\cup(a_i\cap b_{j+1})=a_{i,j+1}\,,$$

applying (1), (2) and Theorem 4, we have that $a_{i,j+1}$ is $a_{i,j}/n_0$-modular. Similarly $b_{i+1,j}$ is $b_{i,j}/n_0$-modular.

(ii) Proof of projectivity and isomorphism:
Applying (1), (3) and Theorem 1, we get

$$a_{i,j}/a_{i,j+1}\cong a_i\cap b_j/(a_i\cap b_{j+1})\cup(a_{i+1}\cap b_j).$$

Similarly

$$b_{i,j}/b_{i+1,j}\cong a_i\cap b_j/(a_i\cap b_{j+1})\cup(a_{i+1}\cap b_j).$$

Hence

$a_{i,j}/a_{i,j+1}$ and $b_{i,j}/b_{i+1,j}$ are projective and isomorphic.

This complete the proof.

Remark 1. If there exists an unrefined m/n-modular chain on n_0, then we get the Jordan-Hölder Theorem.

Remark 2. Let L be a modular lattice. Then Theorem 6 is the usual Schreier's Theorem for L.

Remark 3. Let L be a lattice with the following condition: If $x\cup y$ covers x and y, then $x\cap y$ is covered by x and y. Then we get the Jordan-Hölder Theorem for any finite dimensional interval m/n.

Because, if $x, y\in m/n$ and x covers y then y is x/n-modular, therefore, this is immediate by Theorem 6.

Hence, if L is lower semi-modular, then the Jordan-Hölder Theorem for L is a special case of this remark.