99. *On the Classification of Symmetric Fuchsian Groups of Genus Zero*

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1. Let $\{\alpha_i\}$ $(i=0,1,2,\dots)$ be a finite or an enumerable number of circular open arcs in the unit circle $|z|<1$ which are orthogonal to the circumference $|z|=1$ and disjoint each others in $|z|<1$ and let D_0 be the simply connected domain in $|z|<1$ bounded by $\{\alpha_i\}$ $(i=0,1,2,\dots)$ and the closed set E on $|z|=1$. If $\tilde{D}_0$ is the reflection of D_0 with respect to an arc of $\{\alpha_i\}$, say α_0, then the domain $D_0 \cup \alpha_0 \cup \tilde{D}_0$ is a fundamental domain of a symmetric Fuchsian or fuchsoid group $\mathfrak{G}$ without any elliptic transformation and of genus zero. Conversely, such a group has a fundamental domain as stated above.

We denote by $\{\tilde{\alpha}_i\}$ $(i=0,1,2,\dots)$ the boundary arcs of D_0 corresponding to $\{\alpha_i\}$ $(i=0,1,2,\dots)$ by $\mathfrak{G}$. $\tilde{\alpha}_0$ is identical to α_0. Identifying the equivalent points on α_i and $\tilde{\alpha}_i$ $(i=1,2,\dots)$, we get an open Riemann surface $\hat{D}$. This surface $\hat{D}$ can be decomposed by a relative boundary C into two parts D and $\tilde{D}$, each one of which is the image of the other by an indirectly conformal mapping. And $D\cup C$ (or $\tilde{D}\cup C$) is comformally equivalent to $D_0 \cup \bigcup_{i=0}^{\infty} \alpha_i$ (or $\tilde{D}_0 \cup \bigcup_{i=0}^{\infty} \tilde{\alpha}_i$).

2. We state here some notations. Let HB or HD be the class of single-valued harmonic functions bounded or Dirichlet bounded in a region. If there exists no non-constant function of HB (or HD) in D_0 which equals to zero on $\Gamma=\bigcup_{i=0}^{\infty}\alpha_i$, then we may say that D_0 belongs to the class SO_{HB} (or SO_{HD}). If any function of HB in D_0, whose normal derivative vanishes at every point on Γ, reduces to a constant, we say that D_0 belongs to the class NO_{HB}.

Further we denote by O_G the class of Riemann surfaces with null boundary and by O_{AB} (or O_{AD}) the class of Riemann surfaces on each of which there exists no non-constant single-valued bounded (or Dirichlet bounded) analytic function.

3. Ullemar (=Uskila[7) 8)]) classified the symmetric Fuchsian or fuchsoid groups $\mathfrak{G}$ without any elliptic transformation and of genus zero according to the existence of a certain kind of automorphic functions for $\mathfrak{G}$. More precisely, $\mathfrak{G}$ belongs to positive type or null type with respect to bounded (or Dirichlet bounded) automorphic functions according to whether in $|z|<1$ there exists a non-

constant single-valued bounded (or Dirichlet bounded) automorphic function or not.

It is obvious that $\mathfrak{G}$ belongs to null type with respect to bounded (or Dirichlet bounded) automorphic functions if and only if $\hat{D} \in O_{AB}$ (or $\in O_{AD}$).

On the other hand, Ohtsuka[4] proved that $\hat{D} \in O_G$ if and only if every single-valued non-negative subharmonic function on $\hat{D}$ reduces to a constant. Hence we may say that the group $\mathfrak{G}$ belongs to positive type or null type with respect to non-negative subharmonic automorphic functions according to whether $\hat{D} \notin O_G$ or $\hat{D} \in O_G$.

The following is known (Ullemar[7][8], Kuroda[2]).

Proposition 1.

$\hat{D} \in O_G$ *if and only if* $D_0 \in NO_{HB}$.

$\hat{D} \in O_{AB}$ *if and only if* $D_0 \in SO_{HB}$.

$\hat{D} \in O_{AD}$ *if and only if* $D_0 \in SO_{HD}$.

Mapping D_0 in the unit circle $|w|<1$, then we have the closed set e lying on $|w|=1$ and corresponding to E. Then we can obtain the following without any difficulty. (Cf. Kametani[1], Sario[6], Kuroda[2].)

Proposition 2.

$\hat{D} \in O_G$ *if and only if the set e is of logarithmic capacity zero.*

$\hat{D} \in O_{AB}$ *if and only if the set e is of measure zero.*

$\hat{D} \in O_{AD}$ *if and only if the span of the complementary domain of the set e is zero.*

4. Here we shall give a sufficient condition for $\hat{D} \in O_G$. Let θ_r be the part of the circumference $|z|=r\,(<1)$ contained in D_0. We denote by $l(r)$ the length of θ_r. Then we have

Theorem 1. *If the integral*

$$\int^1 \frac{dr}{l(r)}$$

diverges, then $\hat{D} \in O_G$.

Proof. By Proposition 1, it is sufficient to prove that $D_0 \in NO_{HB}$.

Let u be a function of HB in D_0 whose normal derivative $\frac{\partial u}{\partial \nu}$ vanishes at every point on Γ. If we denote by $D(r)$ the Dirichlet integral of u taken over the subdomain D_r of D_0 which is the common part of D_0 and $|z|<r$, then it is easy to see that

$$D(r) = \int_{\theta_r} u \frac{\partial u}{\partial r} r\, d\theta\,, \qquad z = re^{i\theta}\,, \tag{1}$$

for $\frac{\partial u}{\partial \nu}$ equals to zero on Γ. By the Schwarz inequality, we obtain

$$D^2(r) \leqq M^2 \int_{\theta_r} r\,d\theta \int_{\theta_r} \left(\frac{\partial u}{\partial r}\right)^2 r\,d\theta,$$

provided that $|u|<M$. Since

$$l(r)=\int_{\theta_r} r\,d\theta \quad \text{and} \quad \int_{\theta_r} \left(\frac{\theta u}{\theta r}\right)^2 r\,d\theta \leqq \frac{dD(r)}{dr},$$

we get easily

$$\int_{r_0}^{r} \frac{dr}{l(r)} \leqq \frac{1}{D(r_0)} - \frac{1}{D(r)} \leqq \frac{1}{D(r_0)}.$$

Hence the function u must be a constant, if the integral

$$\int^1 \frac{dr}{l(r)}$$

diverges. Thus, under the condition of our theorem, $D_0 \in NO_{HB}$. Therefore, we arrive at the required.

5. From F.-M. Riesz' theorem and Proposition 2, we can easily see the following.

Theorem 2. *Put* $\theta(r)=\int_{\theta_r} d\theta$. *Then* $\hat{D} \in O_{AB}$ *if and only if* $\lim_{r\to 1} \theta(r)=0$.

6. Now we shall give a sufficient condition in order that $\hat{D} \in O_{AD}$. θ_r is constructed by a finite number of circular arcs θ_r^i ($i=1,\dots,n_r$) which are disjoint each others. We denote by $\lambda_i(r)$ the length of θ_r^i and put

$$\lambda(r)=\operatorname*{Max}_i \lambda_i(r).$$

Then we can prove the following.

Theorem 3. *If the integral*

$$\int^1 \frac{dr}{\lambda(r)}$$

diverges, then $\hat{D} \in O_{AB}$.

Proof. Let u be a harmonic function which equals to zero on Γ. If $D(r)$ is the Dirichlet integral of u taken over D_r, the formula (1) holds good again. By Wirtinger's inequality, we get

$$(2) \qquad \int_{\theta_r^i} u^2 r\,d\theta \leqq \frac{(\lambda_i(r))^2}{\pi^2} \int_{\theta_r^i} \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 r\,d\theta \leqq \frac{(\lambda(r))^2}{\pi^2} \int_{\theta_r^i} \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 r\,d\theta.$$

On the other hand, using the Schwartz inequality, we obtain

$$(3) \qquad \left(\int_{\theta_r^i} u\frac{\partial u}{\partial r} r\,d\theta\right)^2 \leqq \int_{\theta_r^i} u^2 r\,d\theta \int_{\theta_r^i} \left(\frac{\partial u}{\partial r}\right)^2 r\,d\theta,$$

whence, by (2) and (3), it follows that

$$\int_{\theta_r^i} u\frac{\partial u}{\partial r} r\,d\theta \leqq \frac{\lambda(r)}{\pi} \sqrt{\int_{\theta_r^i} \left(\frac{\partial u}{\partial r}\right)^2 r\,d\theta \int_{\theta_r^i} \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 r\,d\theta} \leqq \frac{\lambda(r)}{2\pi} \int_{\theta_r^i} \left[\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2\right] r\,d\theta.$$

Summing up these inequalities for $i=1,\dots,n_r$, we have

$$D(r) \leqq \frac{\lambda(r)}{2\pi} \frac{dD(r)}{dr}$$

or

$$\frac{dr}{\lambda(r)} \leqq \frac{1}{2\pi} \frac{dD(r)}{D(r)}.$$

By integrating both sides, we get

$$\int_{r_0}^{r} \frac{dr}{\lambda(r)} \leqq \frac{1}{2\pi} \log \frac{D(r)}{D(r_0)}.$$

Therefore, if the integral

$$\int^1 \frac{dr}{\lambda(r)}$$

diverges, then $\lim_{r\to 1} D(r)=\infty$ or $D(r_0)=0$. Thus, under the condition of our theorem, the harmonic function which equals to zero on Γ, has not a finite Dirichlet integral or reduces to a constant, and hence D_0 has to belong to SO_{HD}. From Proposition 1, we get the assertion.

7. *Remark.* Theorems 1 and 3 are similar to the results obtained by Laasonen[3] and Sario[5] respectively. They considered the case when D_0 is a fundamental domain, containing the origin of the unit circle, of any Fuchsian or fuchsoid group.

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