96. On Selberg's Function

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1. In a recent paper, A. Selberg has achieved an elementary proof of Dirichlet's theorem about primes in an arithmetic progression[5] (numbers in square brackets refer to the references at the end of this note), and his proof is based upon the following Selberg's Inequality:

$$(1)\qquad \frac{x}{k}V(x)=\sum_{p\leq x,\ p\equiv\lambda(k)}\log^2 p+\sum_{pq\leq x,\ pq\equiv\lambda(k)}\log p\log q+O(x),$$

where

$$(2)\qquad V(x)=\sum_{d\leq x,\ (d,k)=1}\frac{\mu(d)}{d}\log^2\frac{x}{d}=\frac{2}{\varphi(k)}x\log x+O(x).$$

For every positive integer k, $\mu(k)$ and $\varphi(k)$ are the Möbius function and the Euler function respectively. p, q are primes and $(k, l)=1$.

We shall give in this note the generalized forms of (1) and (2) (Theorems 1, 2 and 3). Our method is based upon Selberg's original papers[5][6], and Shapiro's[7]. The umbral calculus is very effective in our description of the calculations and results[1]. The results of our previous paper[2] are used here without proofs.

2. Preliminary Lemmas and Notions

Lemma 1. For every integers k and i, the number theoretic function $[k]^i\geqq 0$ with the following initial conditions: $k\geqq 0$, $k\geqq i$, $[0]^i=1$ for $i=0,1$, $1/|i|!$ for $i<0$ and $[k]^i=0$ for $k<i$, is defined by the recurrence formula $[k]^i=[k-i]^i+i[k-1]^{i-1}$. Then, we get $[k]^i=k!/(k-i)!$ $(i\lesseqgtr 0)$. $[k]^i$ $(i\gtreqless 0)$. is said the factorial polynomial in k degree i.

Lemma 2.

$$\sum_{i=l+m}^{k}(-1)^i[i]^m\binom{k}{i}\binom{i-m}{l}=\begin{cases}0, & \text{for } k\neq l+m,\\ (-1)^k[k]^{k-l}, & \text{for } k=l+m,\end{cases}$$

where $\binom{k}{i}=[k]!/i!$, $k\geqq i\geqq 0$ is the binomial coefficient.

Lemma 3. λ_n is a partition of n and if there are m_1 parts equal to 1, m_2 parts equal to 2, m_3 parts equal to 3, etc., then the partition may be written as[4] $\lambda_n=(1^{m_1}2^{m_2}3^{m_3}\ldots)$, $m_i\geqq 0$, and we put $m=\sum_{i=1}^{m}m_i$, $p(\lambda_n)=m!/m_1!\ m_2!\ \ldots\ m_n!=\binom{m}{m_1,m_2,\cdots m_n}$. We associate a monomial $M(\lambda_n, x)=M(\lambda_n,x_1,\ldots,x_n)=x_1^{m_1}x_2^{m_2}\ldots x_n^{m_n}$ with a partition λ_n. Put $A^n=A^n(x)=A^n(x_1,x_2,\ldots,x_n)=\sum_{\lambda_n}p(\lambda_n)M(\lambda_n,x)$, then, we have $A^n=\sum_{j=1}^{n}x_jA^{n-j}$.

If the number theoretic function $f(j,k)$ is defined by the following initial conditions: $f(l,l-1)=l$ for $l\geqq 2$, $f(l,i)=0$ for $i\geqq l$, and $f(1,0)=0$, and the recurrence formula

$$f(j,k) = k\sum_{i=j+1}^{k-1}(-1)^{k-1}\binom{k-1}{i}x_{k-i}f(j,i), \qquad 1\leqq j\leqq k-2,$$

then $f(j,k)=(-1)^{k-1-j}[k]^{k-j}A^{k-1-j}(y)$, where $y_j=x_j/(j-1)!$, $1\leqq j\leqq n$.

Proof. (Y. Eda[2]), lemmas 3, 4, 5 and 6.

We must develop the Bell's umbral calculus a little, but it will be reserved for another occasion for further particulars. The functional umbra $X(a)$ is denoted by the one-rowed matrix $X(a)\equiv (X^0(a), X^1(a),\dots,X^i(a),\dots)=(X^i(a))$, where $X^i(a)$ $(i=0,1,2,\dots)$ are scalars (real numbers) or any number theoretic function of a, and the $(n+1)$-th element of $X(a)$ is denoted by $(X(a))_n$. $X(a)=(X(a), X(a),\dots)$ is called a constant umbra. Equality of umbra is matric equality and scalar product, $\alpha X(a)$ of α (scalar) and $X(a)$ is $\alpha X(a)\equiv(\alpha X^n(a))$. The umbral sum (difference) of $X(a)$ and $Y(a)$ is $X(a) \overset{\oplus}{\ominus} X(a)\equiv(X^n(a)\pm Y^n(a))$. If $X(a), Y(a),\dots, Z(a)$ are t distinct umbrae, $[X(a)\oplus Y(a)\oplus\cdots\oplus Z(a)]^i$ denotes the scalar $P^i(a)=\sum_{i_1+\dots+i_t=i}\binom{i}{i_1,\dots,i_t}X^{i_1}(a)\dots Z^{i_t}(a)$. Note that exponents and suffixes 0, 1 are to be indicated precisely in the same way as exponents and suffixes >1. If an umbra $X(a)$ in our region is transformed into another umbra $Y(a)$ by a mapping T, we write this fact as follows: $TX(a)=Y(a)$, $Y^i(a)=TX^i(a)$. T is called an umbral operator. A scalar in the scalar product is an umbral operator. The operator product T_2T_1 of T_1 and T_2 is defined by $T_2T_1X(a)=T_2(T_1X(a))$. If we put $T\equiv\sum_{a\in\Omega}F(a)$ and $TX(a)=\sum_{a\in\Omega}F(a)X(a)=(\sum_{a\in\Omega}F^i(a)X^i(a))$, then T is treated as an umbral operator and we call this a $\sum$-operator. If T is T-invarient i.e. $TA=A$, then $T[X(a)\oplus A]^n=[TX(a)\oplus A]^n$. I-operator of an umbra is defined by the formula $IX(a)=Y(a)$, $Y^n(a)=\dfrac{1}{n+1}X^{n+1}(a)$. If $F(x)=(F^i(x))$ and $\lim_{x\to a}F^i(x)=F^i(a)$, $(F(a)=(F^i(a)))$, then we write $\lim_{x\to a}F(x)=(\lim_{x\to a}F^i(x))$. And, still more, if $F^i(x)=f^i(x)+O(R^i(x))$, then we must write $F(x)=(f(x)\oplus O(R(x))$.

Now, we define some $\sum$-operators as follows: $G_x=\sum_{d\leq x}1$, $D_a=\sum_{d\mid a}1$, $E_{x,a}=\sum_{d\leq x,\,(d,a)=1}1$, $A_{x,a,\lambda}=\sum_{d\leq x,\,d\equiv\lambda(a)}1$, $\bar{G}_x=G_x\mu(d)$, $\bar{D}_a=D_a\mu(d)$, $\bar{E}_x=E_{x,a}\mu(d)$, $\bar{A}_x=A_x\mu(d)$, $G_x^*=G_x\dfrac{1}{d}$, $D_a^*=D_a\dfrac{1}{d}$, $E_x^*=E_x\dfrac{1}{d}$, $A_x^*=A_x\dfrac{1}{d}$, $U(x)=\bar{G}^*L(d)$, $K(a)=\bar{D}^*L(d)$, $V(x)=\bar{E}^*L(d)$, $W(x)=\bar{A}^*L(d)$, where $L(x)$ is the log i.e. $L^i(x)=\log^i x$. Then we have (as an operator product), $E_x^*=G_x^*D_{(d,a)}\mu(d)$. If we put $\Theta(a,x)=\bar{D}L\left(\dfrac{x}{d}\right)$, then

$$(3)\qquad A_{x,a,\lambda}\Theta(a,x)=\frac{\lambda}{a}\bar{E}L\left(\frac{x}{d}\right)\oplus O(x)=\frac{x}{a}V(x)\oplus O(x).$$

If we use the $\sum$-operator as an ordinary summation, there is no confusion in our calculation. For example, $\bar{D}^*1=\frac{\varphi(a)}{a}=K^0(a)=K$, $\bar{E}^*\cdot 1=V^0(x)=V=O(1)$, (E. Landau[3], p. 568), $V^k(x)=V^k=O(L^{k-1}(x))$, (Y. Eda[2], lemma 11).

Lemma 4. N denotes the number of different prime factors of a and we put $\Theta(a,x)=\bar{D}L\left(\frac{x}{d}\right)$, $n=p_1^{m_1}p_2^{m_2}\ldots p_N^{m_N}$, then we get $\Theta(a,x)=k!\prod_{i=1}^{k}L(p_i)$ for $N=k$, $\Theta^k(a,x)=0$ for $N>k$. And we get

$$A_{x,a,\lambda}\Theta^k(a,x)=\sum_{j=0}^{k}(-1)^j\Big[\sum_{l=0}^{N-1}(-1)^l\binom{k}{a_1\ldots a_l}\sum_{p_1\cdots p_l}\vartheta(\alpha_1,\ldots,\alpha_l;a_1\ldots a_l;x)\Big]L^{k-j}(x),$$

where

$$\vartheta(\alpha_1\ldots\alpha_l,a_1\ldots a_l,x)=\sum_{p_1^{\alpha_1}\cdots p_l^{\alpha_l}\le x,\text{ and }\equiv\lambda(\mathrm{mod}\ a)}L^{a_1}(p_1)\ldots L^{a_l}(p_l)$$

and $d\equiv x$ (mod a). (Y. Eda[2]), lemma 2 and lemma 14.

Lemma 5.

$$G_x^*L(d)=IL(x)\oplus C\oplus O\left(\frac{L(x)}{x}\right),$$

where $C=\lim_{x\to\infty}(G_x^*L(d)\ominus IL(x))$, which is called the Euler umbra and C° denotes the ordinary Euler constant.

Proof. (E. Landau[3]), 27, Hilfssatz.

Lemma 6. (Stirling's formula) $G_xL(d)=x([L(x)\ominus[k]]^k)$.

Proof. (Y. Eda[2]), lemma 7.

Lemma 7. $IV(x)=O(L(x))$.

Proof. See Y. Eda[2], lemma 11.

3. Proof of the Theorems

Lemma 8.

$$E_{a,x}^*L(d)=K^0IL(x)\ominus IK\oplus([K\oplus C)\oplus O\left(\frac{L(x)}{x}\right).$$

Proof.

$$E_{a,x}^*L(d)=G_x^*L(d)D_{(n,a)}=([K_a^0Z\oplus 1])=U(x),$$

where

$$K_a^0Z^i=K_a^0G_{x/\delta}^*L^i(t)L^{k-i}(\delta)$$
$$=[i]^{-1}\sum_{j=0}^{i+1}(-1)^{i+1-j}\binom{i+1}{j}K_a^{k+1-j}L^j(x)+K_a^{k-i}C^i+O\left(\frac{L^k(x)}{x}\right),$$

and so, we obtain from lemma 2,

$$U^k(x)=[k]^{-1}K^0L^{k+1}(x)-[k]^{-1}K^{n+1}+[K\oplus C]^k+O\left(\frac{L^k(x)}{x}\right).$$

Lemma 9. $1\leq\lambda\leq a-1$,

$$A_{x,a,\lambda} L(d) = \frac{x}{a}([L(x) \ominus k]^k) \oplus O(L(x)) = \frac{1}{a} G_x L(n) \oplus O(L(x)).$$

Proof. From lemma 6, we get

$$AL^k(d) = \sum_{i=0}^{k} \binom{k}{i} L^{k-i}(a) \left[L\left(\frac{x}{a}\right) \ominus [i] \right]^i + O(L^k(x))$$
$$= \frac{x}{a} \left[L(x) \ominus [k] \right]^k + O(L^k(X)).$$

Lemma 10.

$$E_{x,a} A_{\frac{x}{d},\lambda,a} L(d') = \frac{x}{a} I\left(\left[V(x) - [k]^k \right] \right) \oplus O(L(x)).$$

Lemma 11.

$$E^*_{a,x} L\left(\frac{x}{d}\right) = I([L(x) \ominus K_a]) \oplus ([L(x) \ominus [K \oplus C]) \oplus O\left(\frac{L(x)}{x}\right).$$

Proof.

$$E^*_{a,x} L\left(\frac{x}{d}\right) = E^*([L(x) \ominus L(d)]) = ([L(x) \ominus E^* L(d)]).$$

From lemma 7, we get the result.

Theorem 1.

$$I\left([V \ominus K] \right) \oplus \left(\left[V \ominus [K \oplus C] \right] \right) \ominus L(x) = O(1).$$

Proof. If $F(x)$ is any real valued functional umbra, defined for all real $x > 0$, and $G(x)$ is defined by $G(x) = E'_{x,a} F\left(\frac{x}{d}\right)$, then $F(x) = \bar{E}_{x,a} G\left(\frac{x}{d}\right)$. Put $F(x) = x(L(x))$ in this lemma, then

$$G(x) = x\left(I[L(x) \ominus K] \right) \oplus \left(\left[L(x) \ominus [K \oplus C] \right] \right) \oplus O(L(x)),$$

and we get our result.

Remark. We can solve this equality generally. In other words, $V(x)$ is represented by a permanent[4], whose elements are $\lambda L^i(x)$, (λ is scalar), ((Y. Eda[2]) lemma 12 and lemma 13).

Theorem 2.

$$IV(x) = \left([L(x) \ominus B] \right) \oplus O(1)$$

where $B^l = [k]^l A^l(y)$, (A^l in lemma 3), $y_j = \Lambda^j / K(j-1)$, and $\Lambda = IK \ominus ([K \oplus C])$.

Proof. From our theorem 1, we get

$$K^0 IV(x) = \left([V(x) \ominus \Lambda] \right) \oplus L(x) \oplus O(1).$$

Now, assume $V^i(x) = \sum_{j=1}^{i-1} \gamma_i^j L(x) + O(1)$, then we get the recurrence formula and initial conditions for γ_i^j, and we have easily from the lemma 3 our desired form.

Theorem 3. (Selberg's Inequality)

$$A_x \Theta(a,x) = \frac{x}{a} V(x) \oplus O(x),$$

where $\Theta(a,x)$ and $V(x)$ are given by lemma 4, lemma 5 and theorem 2.

Proof. We have the result immediately from (3), lemma 5 and theorem 2.[6]

References

1) E. T. Bell: Postulational Bases for the Umbral Calculus, Amer. J. Math., **62**, 717–724 (1940).

2) Y. Eda: On the Selberg's Inequality, Sci. Rep. Kanazawa Univ., **2**, 7–13 (1953).

3) E. Landau: Handbuch der Lehre von der Verteilung der Primzahlen, Teubner, **2** (1909).

4) D. E. Littlewood: The Theory of Group Characters and Matrix Representations of Groups, Oxford, 2nd ed. (1940).

5) A. Selberg: An Elementary Proof of Dirichlet's Theorem about Primes in an Arithmetic Progression, Ann. Math., **50**, 297–304 (1949).

6) A. Selberg: An Elementary Proof of the Prime Number Theorem, Ann. Math., **50**, 305–313 (1949).

7) H. N. Shapiro: On Primes in Arithmetic Progressions. I, Ann. Math., **52**, 485–497 (1951).