

114. On a Proof of a Theorem of Rosenberg

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Recently, A. Rosenberg³⁾ solved affirmatively a conjecture of M.A. Neumark²⁾ as follows:

Theorem. *A C^* -algebra, having unique irreducible representation and acting irreducibly on a separable Hilbert space, is the algebra of all completely continuous operators.*

The purpose of the present note is to give an alternative proof of the Theorem, which may be simpler than the original Rosenberg's and free from the results of M.A. Neumark^{*)}.

Let H be a separable Hilbert space on which a C^* -algebra A in question acts irreducibly, and C be the C^* -algebra of all completely continuous operators on H . Then the proof of Theorem is divided into the following two Lemmas:

Lemma 1. *Either A coincides with C , or $A \cap C = 0$.*

Lemma 2. *Let B be a C^* -algebra generated by A and C , then B also satisfies the assumptions of Theorem.*

Proof of Lemma 1. Suppose that A does not coincide with C . If $A \cap C = D$ is not 0, then D is a proper two-sided closed ideal of A . Whence A/D is a non-trivial C^* -algebra, and has therefore an irreducible representation, which is different from that of A . Thus we arrive at a contradiction.

We remark, in addition, that we can show the simplicity of A by the same arguments as above.

Proof of Lemma 2. To complete the proof of Lemma 2, it is sufficient to show that every pure state of B is a wave function (considering B as an algebra on H).

Let π be a pure state of B , then without loss of generality we can assume that π is a pure state of the full operator algebra on H . (Cf. I. E. Segal⁴⁾; Lemma 2²⁾.) Moreover, if π is not a wave function, then by the last theorem of J. Dixmier¹⁾, π vanishes on C , and, by the assumption on A , π is a non-pure state of A . Therefore, the kernel D of an irreducible representation of B induced by π contains C , and $A \cap D = 0$ since A is simple.

Now, if we denote by E the algebraic sum of A and C , that is,

$$E = \{a + c \mid a \in A, c \in C\},$$

^{*)} Original paper of M.A. Neumark is unavailable to the authors who knew it by Rickart's review in the "Mathematical Reviews".

then E is a dense subring of B , and $E \cap D = C$. Hence the image of $E/D = E/C = A$ must be dense in the image of B/D .

Let H' be a representation space of B induced by π , and ξ be an element of H' , then $\xi [B/D]^{**}$ is dense in H' , therefore $\xi [E/D]^{**} = \xi A^{**}$ is also dense in H' .***) This shows that the representation A^{**} of A induced by π is irreducible, therefore, by the hypothesis of A , π must be a wave function. Thus we obtain a contradiction.

References

- 1) J. Dixmier: Les fonctionnelles linéaires sur l'ensemble des opérateurs bornés d'un espace de Hilbert, *Ann. Math.*, **51**, 387-408 (1950).
- 2) M. A. Neumark: On a problem of the theory of rings with involution, *Uspehi Mat. Nauk*, **6**, 160-164 (1951).
- 3) A. Rosenberg: The number of irreducible representations of simple rings with no minimal ideals, *Amer. Jour. Math.*, **75**, 523-530 (1953).
- 4) I. E. Segal: Postulates for general quantum mechanics, *Ann. Math.*, **48**, 931-948 (1947).

) a^{} denotes the operator on H' by which the element a is represented.