112. On Completeness of Uniform Spaces

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Let R be an abstract space. For a system of mappings $\mathfrak{a}_\lambda$ of R into uniform spaces $S_\lambda\,(\lambda \in \Lambda)$, the weakest uniformity on R for which all $\mathfrak{a}_\lambda\,(\lambda \in \Lambda)$ are uniformly continuous, is called the *weak uniformity of* R by $\mathfrak{a}_\lambda\,(\lambda \in \Lambda)$. Concerning the completeness of the weak uniformity we have[1)]

Theorem I. *Let the uniformities* $\mathfrak{U}_\lambda$ *of* $S_\lambda\,(\lambda \in \Lambda)$ *be separative and complete. In order that the weak uniformity of* R *by a system of mappings* $\mathfrak{a}_\lambda$ *of* R *into* $S_\lambda\,(\lambda \in \Lambda)$ *be complete, it is necessary and sufficient that for a system of points* $x_\lambda \in S_\lambda\,(\lambda \in \Lambda)$ *if*

$$\prod_{\nu=1}^{n} \mathfrak{a}_{\lambda_\nu}^{-1}\left(U_{\lambda_\nu}(x_{\lambda_\nu})\right) \neq 0$$

for every finite number of elements $\lambda_\nu \in \Lambda$ *and* $U_{\lambda_\nu} \in \mathfrak{U}_{\lambda_\nu}(\nu=1,2,\ldots,n)$, *then we can find a point* $x \in R$ *for which* $\mathfrak{a}_\lambda(x)=x_\lambda$ *for every* $\lambda \in \Lambda$.

The purpose of this paper is to give some generalization of this Theorem I and its applications.

I

For a uniform space R with uniformity $\mathfrak{V}$, a system of mappings $\mathfrak{a}_\gamma\,(\gamma \in \Gamma)$ of R into a uniform space S with uniformity $\mathfrak{U}$ is said to be *equi-continuous*, if for any $U \in \mathfrak{U}$ we can find $V \in \mathfrak{V}$ such that

$$\mathfrak{a}_\gamma(V(x)) \subset U(\mathfrak{a}_\gamma(x)) \qquad \text{for every } x \in R \text{ and } \gamma \in \Gamma.$$

With this definition we have

Theorem II. *Let the uniformity* $\mathfrak{U}_\lambda$ *of* $S_\lambda\,(\lambda \in \Lambda)$ *be separative and complete. For a double system of mappings* $\mathfrak{a}_{\gamma,\lambda}$ *of an abstract space* R *into* $S_\lambda\,(\gamma \in \Gamma_\lambda, \lambda \in \Lambda)$, *there exists the weakest uniformity on* R *for which* $\mathfrak{a}_{\gamma,\lambda}\,(\gamma \in \Gamma_\lambda)$ *is equi-continuous for every* $\lambda \in \Lambda$, *and in order that this uniformity on* R *be complete, it is necessary and sufficient that for a system of points* $x_{\gamma,\lambda} \in S_\lambda\,(\gamma \in \Gamma_\lambda, \lambda \in \Lambda)$ *if*

$$\prod_{\nu=1}^{n} \prod_{\gamma \in \Gamma_{\lambda_\nu}} \mathfrak{a}_{\gamma,\lambda_\nu}^{-1}\left(U_{\lambda_\nu}(x_{\gamma,\lambda_\nu})\right) \neq 0$$

for every finite number of elements $\lambda_\nu \in \Lambda$ *and* $U_{\lambda_\nu} \in \mathfrak{U}_{\lambda_\nu}(\nu=1,2,\ldots,n)$, *then we can find a point* $x \in R$ *such that*

$$x_{\gamma,\lambda} = \mathfrak{a}_{\gamma,\lambda}(x) \qquad \textit{for all } \gamma \in \Gamma_\lambda, \lambda \in \Lambda.$$

1) H. Nakano: Topology and linear topological spaces, Tokyo Math. Book Ser. II, Tokyo (1951), § 35 Theorem 8. In the present paper we make use of terminologies and notations in this book. This book will be denoted by TLTS.

In order to prove this Theorem II, we shall define power of a uniformity. Let S be a uniform space with uniformity $\mathfrak{U}$. For another abstract space Λ, considering every system $x_\lambda \in S(\lambda \in \Lambda)$ a point $(x_\lambda)_{\lambda\in\Lambda}$, we obtain a space, which is called the *power* of S by Λ and denoted by S^Λ. For each $U \in \mathfrak{U}$, putting

$$U^\Lambda(x_\lambda)_{\lambda\in\Lambda}=\{(y_\lambda)_{\lambda\in\Lambda}: \ y_\lambda \in U(x_\lambda) \text{ for every } \lambda \in \Lambda\},$$

we obtain a connector U^Λ in S^Λ. Furthermore we see easily that there exists uniquely a uniformity on S^Λ of which $U^\Lambda(U \in \mathfrak{U})$ is a basis. This uniformity on S^Λ is called the *power* of $\mathfrak{U}$ by Λ and denoted by $\mathfrak{U}^\Lambda$. With this definition we can prove easily that if $\mathfrak{U}$ is separative, then $\mathfrak{U}^\Lambda$ also is separative; and if $\mathfrak{U}$ is complete, then $\mathfrak{U}^\Lambda$ also is complete.

For a system of mappings $\mathfrak{a}_\lambda(\lambda \in \Lambda)$ of a uniform space R into a uniform space S with uniformity $\mathfrak{U}$, it is evident by definition that $\mathfrak{a}_\lambda(\lambda \in \Lambda)$ is equi-continuous if and only if the mapping $\mathfrak{a}$ of R into the power S^Λ with uniformity $\mathfrak{U}^\Lambda$:

$$\mathfrak{a}(x) = (\mathfrak{a}_\lambda(x))_{\lambda\in\Lambda} \in S^\Lambda \qquad (x \in R)$$

is uniformly continuous. Therefore for a system of mappings $\mathfrak{a}_{\gamma,\lambda}$ of an abstract space R into uniform spaces $S_\lambda(\gamma \in \Gamma_\lambda, \lambda \in \Lambda)$, the weak uniformity of R by the system of mappings $\mathfrak{a}_\lambda$ of R into the uniform spaces $S_\lambda{}^{\Gamma_\lambda}(\lambda \in \Lambda)$:

$$\mathfrak{a}_\lambda(x) = (\mathfrak{a}_{\gamma,\lambda}(x))_{\gamma\in\Gamma_\lambda} \in S_\lambda{}^{\Gamma_\lambda} \qquad (x \in R)$$

is the weakest uniformity on R for which $\mathfrak{a}_{\gamma,\lambda}(\gamma \in \Gamma_\lambda)$ is equi-continuous for every $\lambda \in \Lambda$. Therefore we conclude Theorem II immediately from Theorem I.

In Theorem II, if all uniform spaces $S_\lambda(\lambda \in \Lambda)$ coincide with a complete separative uniform space S with uniformity $\mathfrak{U}$, and for a system of points $x_{\gamma,\lambda} \in S(\gamma \in \Gamma_\lambda, \lambda \in \Lambda)$ if

$$\prod_{\nu=1}^{n} \prod_{\gamma\in\Gamma_{\lambda_\nu}} \mathfrak{a}_{\gamma,\lambda_\nu}{}^{-1}(U_\nu(x_{\gamma,\lambda_\nu})) \neq 0$$

for every finite number of elements $\lambda_\nu \in \Lambda$ and $U_\nu \in \mathfrak{U}(\nu=1,2,\ldots,n)$, then $\mathfrak{a}_{\gamma,\lambda}=\mathfrak{a}_{\gamma',\lambda'}$ implies $x_{\gamma,\lambda}=x_{\gamma',\lambda'}$, because $\mathfrak{U}$ is seperative by assumption. Therefore we conclude from Theorem II

Theorem III. *Let $\mathfrak{a}_\lambda(\lambda \in \Lambda)$ be a system of mappings of an abstract space R into a uniform space S with a complete separative uniformity $\mathfrak{U}$. For a system of subsets $\Lambda_\gamma \subset \Lambda(\gamma \in \Gamma)$ there exists the weakest uniformity on R for which $\mathfrak{a}_\lambda(\lambda \in \Lambda_\gamma)$ is equi-continuous for every $\gamma \in \Gamma$, and if $\Lambda= \sum_{\gamma\in\Gamma} \Lambda_\gamma$ and for any $\gamma_1,\gamma_2 \in \Gamma$ we can find an element $\gamma \in \Gamma$ such that $\Lambda_{\gamma_1} \dotplus \Lambda_{\gamma_2} \subset \Lambda_\gamma$, then in order that this weakest uniformity on R be complete, it is necessary and sufficient that for a system of points $x_\lambda \in S(\lambda \in \Lambda)$ if*

$$\prod_{\lambda\in\Lambda_\gamma} \mathfrak{a}_\lambda{}^{-1}(U(x_\lambda)) \neq 0$$

for every $\gamma \in \Gamma$ *and* $U \in \mathfrak{U}$, *then we can find a point* $x \in R$ *such that* $\mathfrak{a}_\lambda(x) = x_\lambda$ *for every* $\lambda \in \Lambda$.

Let $\mathfrak{A}$ be the totality of mappings of an abstract space R into a uniform space S which a complete separative uniformity $\mathfrak{U}$. Every point $x \in R$ may be considered a mapping of $\mathfrak{A}$ into S as $\mathfrak{a}(x) \in S (\mathfrak{a} \in \mathfrak{A})$. For a system of subsets $R_\lambda \subset R (\lambda \in \Lambda)$ there exists by Theorem III the weakest uniformity on $\mathfrak{A}$ for which R_λ is equi-continuous as a system of mappings for every $\lambda \in \Lambda$. This weakest uniformity on $\mathfrak{A}$ is complete[2], because for any system of points $y_x \in S (x \in R)$ there exists obviously $\mathfrak{a} \in \mathfrak{A}$ for which $\mathfrak{a}(x) = y_x$ for every $x \in R$.

A mapping $\mathfrak{a}$ of R into S is said to be *bounded* in a subset $R_0 \subset R$, if the image $\mathfrak{a}(R_0)$ is a bounded set[3] of S. For a uniformity on $\mathfrak{A}$ if R_λ is equi-continuous as a system of mappings of $\mathfrak{A}$ into S, then we see easily by definition that every convergence by a Cauchy system in $\mathfrak{A}$ is a uniform convergence as mappings of R_λ into S. Therefore on the totality of those mappings of R into S which are bounded in R_λ for every $\lambda \in \Lambda$, the weakest uniformity for which R_λ is equi-continuous for every $\lambda \in \Lambda$, is complete. We conclude further that if R is a topological space, then on the totality of those mappings of R into S which are continuous in R_λ by the relative topology for every $\lambda \in \Lambda$, the weakest uniformity for which R_λ is equi-continuous for every $\lambda \in \Lambda$, is complete. Furthermore we obtain likewise that if R is a uniform space, then on the totality of those mappings of R into S which are uniformly continuous in R_λ by the relative uniformity for every $\lambda \in \Lambda$, the weakest uniformity for which R_λ is equi-continuous for every $\lambda \in \Lambda$, is complete.

II

Let R be a linear space and S a linear topological space with linear topology $\mathfrak{V}$. A system of linear operators $T_\lambda (\lambda \in \Lambda)$ on R into S is said to be *bounded*, if the system $T_\lambda x (\lambda \in \Lambda)$ is a bounded set of S for every $x \in R$. For a bounded system of linear operators $T_\lambda (\lambda \in \Lambda)$ on R into S we see easily that there exists uniquely a linear topology on R of which

$$\prod_{\lambda \in \Lambda} \{x : T_\lambda x \in V\} \qquad (V \in \mathfrak{V})$$

is a basis. Furthermore we see easily that the induced uniformity from this linear topology on R is the weakest uniformity on R for

2) This fact was proved by N. Bourbaki, Topologie générale, Vol. 3, Chapter 10, espaces fonctionnels, Paris (1949).

3) TLTS § 32.

which $T_\lambda(\lambda \in \Lambda)$ is equi-continuous. This linear topology on R is obviously convex, if $\mathfrak{V}$ is convex. Therefore, recalling Theorem 5 in TLTS §55, we obtain by Theorem III

Theorem IV. *Let $T_\lambda(\lambda \in \Lambda)$ be a system of linear operators on a linear space R into a linear topological space S with a complete separative linear topology $\mathfrak{V}$. For a system of subsets $\Lambda_\gamma \subset \Lambda(\gamma \in \Gamma)$, if $T_\lambda(\lambda \in \Lambda_\gamma)$ is a bounded system for every $\gamma \in \Gamma$, then there exists uniquely a linear topology on R whose induced uniformity on R is the weakest uniformity for which $T_\lambda(\lambda \in \Lambda_\gamma)$ is equi-continuous for every $\gamma \in \Gamma$. Furthermore if $\Lambda = \sum_{\gamma \in \Gamma} \Lambda_\gamma$ and for any $\gamma_1, \gamma_2 \in \Gamma$ we can find an element $\gamma \in \Gamma$ such that $\Lambda_{\gamma_1} \dotplus \Lambda_{\gamma_2} \subset \Lambda_\gamma$, then in order that this linear topology on R be complete, it is necessary and sufficient that for a system of elements $x_\lambda \in S(\lambda \in \Lambda)$ if*

$$\prod_{\lambda \in \Lambda_\gamma} \{x : T_\lambda x \in V + x_\lambda\} \neq 0$$

for every $\gamma \in \Gamma$ and $V \in \mathfrak{V}$, then we can find an element $x \in R$ for which $T_\lambda x = x_\lambda$ for every $\lambda \in \Lambda$.

Let R be an abstract space and S a linear topological space with a complete separative linear topology $\mathfrak{V}$. For a subset $R_0 \subset R$ a mapping $\mathfrak{a}$ of R into S is said to be *bounded* if the image $\mathfrak{a}(R_0)$ is a bounded set of S. For a system of subsets $R_\lambda \subset R(\lambda \in \Lambda)$, denoting by $\mathfrak{A}$ the totality of those mappings of R into S which are bounded in R_λ for every $\lambda \in \Lambda$, we obtain a linear space $\mathfrak{A}$, defining

$$(\alpha\mathfrak{a} + \beta\mathfrak{b})(x) = \alpha\mathfrak{a}(x) + \beta\mathfrak{b}(x) \qquad (x \in R)$$

for every $\mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$ and real numbers α, β. Furthermore every point $x \in R$ may be considered a linear operator on R into S as $\mathfrak{a}(x) \in S$ $(\mathfrak{a} \in \mathfrak{A})$ and R_λ is a bounded system of linear operators for every $\lambda \in \Lambda$. For a system of elements $x_y \in S(y \in \sum_{\lambda \in \Lambda} R_\lambda)$ if

$$\prod_{y \in R_\lambda} \{\mathfrak{a} : \mathfrak{a}(y) \in V + x_y\} \neq 0$$

for every $\lambda \in \Lambda$ and $V \in \mathfrak{V}$, then $x_y(y \in R_\lambda)$ is a bounded set of S for every $\lambda \in \Lambda$, and hence putting $\mathfrak{a}_0(y) = x_y$ for $y \in \sum_{\lambda \in \Lambda} R_\lambda$ and $\mathfrak{a}_0(y) = 0$ for every other point y, we have $\mathfrak{a} \in \mathfrak{A}$. Therefore we obtain by Theorem IV

Theorem V. *Let R be an abstract space and S a linear topological space with a complete separative linear topology $\mathfrak{V}$. For a system of subsets $R_\lambda \subset R(\lambda \in \Lambda)$ such that for any $\lambda_1, \lambda_2 \in \Lambda$ we can find an element $\lambda \in \Lambda$ for which $R_{\lambda_1} \dotplus R_{\lambda_2} \subset R_\lambda$, denoting by $\mathfrak{A}$ the totality of those mappings of R into S which are bounded in R_λ for every $\lambda \in \Lambda$, we obtain a complete linear topological space $\mathfrak{A}$ such that*

$$(\alpha\mathfrak{a} + \beta\mathfrak{b})(x) = \alpha\mathfrak{a}(x) + \beta\mathfrak{b}(x) \qquad (x \in R)$$

for every $\mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$ and real numbers α, β, and

$$\{\mathfrak{a} : \mathfrak{a}(R_\lambda) \subset V\} \qquad (\lambda \in \Lambda,\ V \in \mathfrak{V})$$

is a basis of $\mathfrak{A}$. *Furthermore if* $\mathfrak{V}$ *is convex, then* $\mathfrak{A}$ *also is convex.*

If R is a linear space and for any $\lambda_1,\lambda_2 \in \Lambda$ and real numbers α,β we can find an element $\lambda \in \Lambda$ such that $\alpha R_{\lambda_1} \times \beta R_{\lambda_2} \subset R_\lambda$, then, denoting by $\mathfrak{A}$ the totality of those linear operators on R into S which are bounded in R_λ for every $\lambda \in \Lambda$ we see easily that

$$\prod_{y \in R_\lambda} \{\mathfrak{a} : \mathfrak{a}(y) \in V + x_y\} \neq 0$$

for every $\lambda \in \Lambda$ and $V \in \mathfrak{V}$ implies

$$x_{\alpha y_1 + \beta y_2} = \alpha x_{y_1} + \beta x_{y_2}$$

for every $y_1, y_2 \in \sum_{\lambda \in \Lambda} R_\lambda$ and real numbers α,β. Therefore we obtain further

Theorem VI. *Let* R *be a linear space and* S *a linear topological space with a complete separative linear topology* $\mathfrak{V}$. *For a system of subsets* $R_\lambda \subset R (\lambda \in \Lambda)$ *such that for any* $\lambda_1,\lambda_2 \in \Lambda$ *and real numbers* α,β *we can find an element* $\lambda \in \Lambda$ *such that* $\alpha R_{\lambda_1} \times \beta R_{\lambda_2} \subset R_\lambda$, *denoting by* $\mathfrak{T}$ *the totality of those linear operators on* R *into* S *which are bounded in* R_λ *for every* $\lambda \in \Lambda$, *we obtain a complete linear topological space* $\mathfrak{T}$ *such that*

$$\{T : TR_\lambda \subset V\} \qquad (\lambda \in \Lambda,\ V \in \mathfrak{V})$$

is a basis of $\mathfrak{T}$. *Furthermore if* $\mathfrak{V}$ *is convex, then* $\mathfrak{T}$ *also is convex.*

This Theorem VI is a generalization of Theorems 1 and 3 in TLTS § 67.