111. On Right-Regular-Ideal-Rings *)

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1. In his paper **) 4) T. Nakayama defined the notion of regularity of modules, which played an important rôle in his Galois theory. In this note we consider a ring in which every non-zero right ideal is right-regular and we call such a ring a right-regular-ideal-ring. To be easily seen, the notion of right-regular-ideal-rings is a generalization of that of simple rings as well as principal-right-ideal-domains ***).

Throughout this paper, except in the last remark, the term "ring" will mean a non-zero ring with an identity, and K will signify a ring. The notation $\cong$ will be used to denote a K-isomorphism between two K-right-modules, unless otherwise specified. Further by minimum and maximum conditions in rings we shall understand those which are related to the right ideals.

Let M be a K-module. If the identity element of K operates as the identity operator for M, then M is called *unitary*. And if a finite generating system $\{u_1, \dots, u_n\}$ of a unitary K-module M is such that $\sum_{i=1}^{n} u_i k_i = 0\ (k_i \in K)$ implies $k_i = 0\ (i=1, \dots, n)$, then we call it an *independent K-basis* of M.

Let M be a unitary K-module, then we shall denote by M^n the direct sum of its n copies written as column vectors. Thus $M \cong K^m$ means that M has an independent K-basis of m elements. On the other hand, we shall denote by ${}^nM$ the direct sum of its n copies written as row vectors. Naturally, ${}^nM$ may be considered as a K_n-module, where K_n denotes the total $n \times n$ matrix ring over K. Hereafter, let ${}^nM$ stand for the K_n-module with the natural K_n-module structure. To be easily verified, $({}^pM)^q$ is K_p-isomorphic to ${}^p(M^q)$, where p, q are natural numbers. From this fact, we can use the notation ${}^pM^q$ instead of ${}^p(M^q)$ or $({}^pM)^q$.

2. A non-zero unitary K-module M is said to be *right-regular* with respect to K if there exist two natural numbers p, q such that $M^p \cong K^q$. And a ring K is called a *right-regular-ideal-ring*

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**) Numbers in brackets refer to the references at the end of this paper.

***) Throughout the paper, a simple ring means a total matrix ring over a division ring. And a principal-right-ideal-domain means an integral domain in which every right ideal is principal.

(abbreviated, *r-r-i-ring*) if every non-zero right ideal in K is right-regular with respect to K.

Let $\mathfrak{r}(\neq 0)$ be a right ideal in an r-r-i-ring K, then $\mathfrak{r}^p \cong K^q$ for some positive integers p, q. Hence $\mathfrak{r}^p$ possesses an independent K-basis of q column vectors $(r_{1i}, \dots, r_{pi})$ $(i=1, \dots, q)$ with every r_{ki} in $\mathfrak{r}$. Then the system $\{r_{1i};\ i=1, \dots, q\}$ is clearly an ideal basis of $\mathfrak{r}$.

Next, let $\mathfrak{a}, \mathfrak{b}$ be two non-zero two-sided ideals in an r-r-i-ring K. From the regularity of $\mathfrak{a}$ there exist two natural numbers p and q such that $\mathfrak{a}^p = v_1K + \cdots + v_qK$ with an independent K-basis $\{v_i\}$ of $\mathfrak{a}^p$. Then we have $(\mathfrak{a}\cdot\mathfrak{b})^p = \mathfrak{a}^p\cdot\mathfrak{b} = v_1\mathfrak{b} + \cdots + v_q\mathfrak{b} \neq 0$, which implies $\mathfrak{a}\cdot\mathfrak{b}\neq 0$. Thus we have

Theorem 1. *An r-r-i-ring is a prime ring with maximum condition. If the minimum condition is assumed, it coincides with a simple ring.*

As an r-r-i-ring K satisfies maximum condition, it can readily be seen that if $M^p \cong K^q$ for a (regular) K-module M then the rational number q/p is an invariant of M, and in this case we may call it the *rank* of M over K.

Now we prove the following principal theorem:

Theorem 2. *Let K be an r-r-i-ring and let M be a right-regular K-module. Then every non-zero K-submodule of M is right-regular too.*

Proof. Let $M^p \cong K^q$ and N be a non-zero submodule of M. Then N is considered as a submodule of M^p with an independent K-basis of q elements. Accordingly, without loss of generality, we may assume that M has an independent K-basis.

Let $M = u_1K + \cdots + u_mK$, where $\{u_i\}$ is an independent K-basis. For $m=1$, our assertion is clear. Now we assume that it is true for $m-1$. Let N be a non-zero K-submodule of M. When N is formed by the linear combinations of $u_1, \dots, u_{m-1}$ only, there is nothing to prove. Hence we may assume that N contains a linear combination $u_1k_1 + \cdots + u_mk_m$ with $k_m \neq 0$. Then all the k's appearing as the coefficients of u_m form a right ideal $\mathfrak{r}(\neq 0)$ in K, and $\mathfrak{r} \cong N - N_0$, where $N_0 (\leqq u_1K + \cdots + u_{m-1}K)$ is the kernel of the homomorphism ρ of N onto $\mathfrak{r}$ defined by $\rho(u_1k_1 + \cdots + u_mk_m) = k_m$. By our induction hypothesis, for some u, v, $N_0^u \cong K^v$. As K is an r-r-i-ring, $\mathfrak{r}^p \cong K^q$ for some p, q, whence $(N-N_0)^p \cong K^q$. It follows therefore $N^{pu} - N_0^{pu} \cong (N-N_0)^{pu} \cong K^{qu}$. Since $N_0^{pu} \cong K^{pv}$, we have eventually $N^{pu} \cong K^{pv+qu}$.

A brief computation shows the following:

Corollary. *Let K be an r-r-i-ring in which every right ideal has the rank not greater than* 1, *and let M be a right-regular K-module with the rank q/p. Then every non-zero K-submodule of K has the* *rank not greater than q/p.*

Remark. The validity of Theorem 2 is suggested by the results of Everett[1)2)]. In fact, we can prove easily that if every right ideal in a ring K has an independent K-basis then K is a principal-right-ideal-domain.

It is further to be noted that if every finitely generated unitary K-module is regular with respect to K then K is a simple ring.

We prove next the following:

Theorem 3. *A ring K is an r-r-i-ring if and only if the total matrix ring K_n is so.*

Proof. Let K_n be an r-r-i-ring and $\mathfrak{r}\,(\neq 0)$ be a right ideal in K. Then $\mathfrak{r}_n$ is a right ideal in K_n. From the regularity of $\mathfrak{r}_n$ there exist two positive integers p, q such that $\mathfrak{r}_n^p$ is K_n-isomorphic to K_n^q. And $\mathfrak{r}_n \cong \mathfrak{r}^{n^2}$, $K_n \cong K^{n^2}$. Hence $\mathfrak{r}^{n^2 p} \cong K^{n^2 q}$.

Conversely, suppose that K is an r-r-i-ring. Let $\mathfrak{R}$ be a right ideal in K_n. Then, as is well known, the set $\mathfrak{c}$ of all column vectors appearing in a fixed column of $\mathfrak{R}$ is a K-module, and moreover, $\mathfrak{R}$ is K_n-isomorphic to ${}^n\mathfrak{c}$. On the other hand, $\mathfrak{c}$ is considered as a submodule of K^n, thus by Theorem 2, $\mathfrak{c}^p \cong K^q$ for some p and q. $\mathfrak{R}^{pn}$ is K_n-isomorphic to ${}^n\mathfrak{c}^{pn}$, ${}^n\mathfrak{c}^{pn}$ is K_n-isomorphic to ${}^nK^{nq}$ and ${}^nK^{nq}$ is K_n-isomorphic to K_n^q. Hence we have that $\mathfrak{R}^{pn}$ is K_n-isomorphic to K_n^q.

Corollary. *Let K be an r-r-i-ring. Then the K-endomorphism ring (considered as a left operator domain) of any right-regular K-module is also an r-r-i-ring.*

Remark. It is the McCoy's view that the radical of general rings may be defined as the intersection of a certain class of prime ideals. From this view-point, we want to give another definition of radicals as follows: In an arbitrary ring K the intersection R of all the two-sided ideals such that the residue class rings modulo them are r-r-i-rings is called the *radical* of K.

Clearly R contains the McCoy's radical[3)] and it coincides with the classical one under the minimum condition.

In case K has an identity, as is well known, every two-sided ideal in K_n is of the form $\mathfrak{a}_n$ with a two-sided ideal $\mathfrak{a}$ in K, and conversely. Since $K_n - \mathfrak{a}_n$ is ring-isomorphic to $(K-\mathfrak{a})_n$, by Theorem 3, $K_n - \mathfrak{a}_n$ is an r-r-i-ring if and only if $K-\mathfrak{a}$ is so This shows that the radical of K_n is R_n.

References

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