110. Unitary Equivalence of Factors of Type III

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F. J. Murray and J. von Neumann[3] investigated that under what conditions the algebraical isomorphism of two factors implies the spatial isomorphism of them. They have many monumental results in the case of types I and II. The object of this paper is to show that the algebraical isomorphism implies the spatial isomorphism in the case of type III.

1. In this paper, we consider only separable Hilbert spaces. Let $\mathfrak{A}$ be a weakly closed self adjoint operator algebra with the identity on a separable Hilbert space $\mathfrak{H}$ and $\mathfrak{A}'$ be the commutor of it. $\mathfrak{A}$ is called a factor if $\mathfrak{A} \frown \mathfrak{A}'$ contains only scalar multiples of the identity. We say that the projections P and Q of $\mathfrak{A}$ are equivalent, written $P \sim Q$, if there exists an element V such that $VV^* = P$ and $V^*V = Q$. A projection P is called finite if $P \sim Q$ and $P \geqq Q$ imply $P = Q$, otherwise infinite. According to F. J. Murray and J. von Neumann[2], a factor $\mathfrak{A}$ will be said of type III if any non-zero projection of $\mathfrak{A}$ is infinite. Let $\mathfrak{A}$ be a factor of type III, then all non-zero projections are equivalent each other. By $[\mathfrak{A}x]$ we mean the closed linear manifold which is a closure of a manifold $[Ax \mid A \in \mathfrak{A}]$ for any vector x in $\mathfrak{H}$. Notice that the projection on $[\mathfrak{A}'x]$ is contained in $\mathfrak{A}$.

A bounded linear functional σ on an operator algebra $\mathfrak{A}$ is called a state if $\sigma(A^*A) \geqq 0$ for any $A \in \mathfrak{A}$ and $\sigma(I) = 1$ where I is the identity of $\mathfrak{A}$. A state σ is called complete provided that $\sigma(A^*A) = 0$ implies $A = 0$. We shall say that a state σ is countably additive if $\sigma(\sum P_n) = \sum (P_n)$ for any sequence $\{P_n\}$ of mutually orthogonal projections. Let x be a vector with unit norm, then it is clear that $\sigma(A) = (Ax, x)$ is a countably additive state.

2. In this section we shall show two lemmas.

Lemma 1. Let $\mathfrak{A}$ be a factor of type III on a Hilbert space $\mathfrak{H}$, then there exists a vector x in $\mathfrak{H}$ such that $[\mathfrak{A}x] = [\mathfrak{A}'x] = \mathfrak{H}$.

Proof. Let x_1 be any vector and P be the projection on $[\mathfrak{A}'x_1]$. Then P is a non-zero projection in $\mathfrak{A}$ and therefore $P \sim I$ since $\mathfrak{A}$ is of type III. Hence, there exists a partially isometric operator V in $\mathfrak{A}$ such that $V^*V = P$ and $VV^* = I$. Put $x_2 = Vx_1$, then

$$[\mathfrak{A}'x_2] = [\mathfrak{A}'Vx_1] = V[\mathfrak{A}'x_1] = \mathfrak{H}.$$

Let Q be the projection on $[\mathfrak{A}x_2]$, then Q is a non-zero projection in $\mathfrak{A}'$. Since $\mathfrak{A}'$ is of type III too, we have $Q \sim I$ in $\mathfrak{A}'$. Therefore, there exists a partially isometric operator W in $\mathfrak{A}'$ whose initial and final projections are Q and I respectively. Put $x = Wx_2$, then $W^*x = x_2$. Therefore,

$$[\mathfrak{A}x] = [\mathfrak{A}Wx_2] = W[\mathfrak{A}x_2] = \mathfrak{H},$$

and

$$[\mathfrak{A}'x] = [\mathfrak{A}'Wx_2] \subset [\mathfrak{A}'x_2] = [\mathfrak{A}'W^*x] \subset [\mathfrak{A}'x].$$

Thus, we have $[\mathfrak{A}'x] = [\mathfrak{A}'x_2] = \mathfrak{H}$. This proves the lemma.

Lemma 2. Let P and Q be projections in a factor $\mathfrak{A}$ which are equivalent each other. Then the contraction of $\mathfrak{A}$ on the range of P is unitary equivalent to the contraction of $\mathfrak{A}$ on the range of Q.

Proof. Let $\mathfrak{M}$ and $\mathfrak{N}$ be the ranges of P and Q respectively, and the contractions of $\mathfrak{A}$ on $\mathfrak{M}$ and $\mathfrak{N}$ will be denoted by $\mathfrak{A}_{\mathfrak{M}}$ and $\mathfrak{A}_{\mathfrak{N}}$ respectively. It is well known that there exists a one-to-one correspondence between $\mathfrak{A}_{\mathfrak{M}}$ and $P\mathfrak{A}P$.

Now we shall prove that $P\mathfrak{A}P$ and $Q\mathfrak{A}Q$ are algebraically isomorphic. By the assumption, there exists an element $V \in \mathfrak{A}$ with $VV^* = P$ and $V^*V = Q$. We shall define a mapping ϕ by

$$\phi(PAP) = V^*PAPV,$$

then ϕ is a mapping from $P\mathfrak{A}P$ into $Q\mathfrak{A}Q$ since

$$V^*PAPAV = V^*VV^*AVV^*V = QV^*AVQ.$$

Moreover, we have

$$Q\mathfrak{A}Q = V^*VV^*VV^*VV^*V = V^*PV\mathfrak{A}V^*PV \subset V^*P\mathfrak{A}PV,$$

and this implies that ϕ is a mapping from $P\mathfrak{A}P$ onto $Q\mathfrak{A}Q$. Let $VPAPV^* = 0$, then

$$0 = V^*VPAPV^*V = PAP,$$

that is, ϕ is an algebraically isomorphic mapping.

Finally, V is a partially isometric operator whose initial and final domains are $\mathfrak{M}$ and $\mathfrak{N}$ respectively. From above considerations, it is clear that $\mathfrak{A}_{\mathfrak{M}}$ is unitary equivalent to $\mathfrak{A}_{\mathfrak{N}}$. This proves the lemma.

Remark. It is excessive for Lemma 2 that $\mathfrak{A}$ is a factor. Lemma 2 is true for any weakly closed self adjoint operator algebra on a (not necessarily separable) Hilbert space.

3. Now we shall prove the following theorem:

Theorem. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be factors of type III on Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively. Moreover, $\mathfrak{A}_1$ is algebraically isomorphic to $\mathfrak{A}_2$, then $\mathfrak{A}_1$ is unitary equivalent to $\mathfrak{A}_2$.

Proof. By Lemma 1, there exists a vector $y_1 \in \mathfrak{H}_1$ such that $[\mathfrak{A}_1'y_1] = \mathfrak{H}_1$. Then $A^1y_1 = 0$ implies $A^1 = 0$ for $A^1 \in \mathfrak{A}_1$. For any $A^1 \in \mathfrak{A}_1$ we put

$$\sigma_1(A^1) = (A^1y_1, y_1),$$

then σ_1 is a complete countably additive state. Now we shall define a complete countably additive state σ_2 on $\mathfrak{A}_2$ as following:

$$\sigma_2(B^2)=\sigma_1(B^1) \text{ for every } B^2\in\mathfrak{A}_1,$$

where B^1 is a corresponding element in $\mathfrak{A}_1$ to B^2. By a theorem due to H. A. Dye[1] [Theorem 1], there exists a non-zero projection P^2 and a vector $y_2\in\mathfrak{H}_2$ which satisfies

$$\sigma_2(P^2A^2P^2)=(A^2y_2, y_2) \text{ for any } A^2\in\mathfrak{A}_2.$$

Let P^1 be a projection in $\mathfrak{A}_1$ which corresponds to P^2. Then, by Lemma 2, $\mathfrak{A}_i$ is unitary equivalent to the contraction $\mathfrak{B}_i$ of $\mathfrak{A}_i$ on the range $\mathfrak{M}_i$ of P^i, since P^i is equivalent to the identity of $\mathfrak{A}_i$ $(i=1,2)$. Therefore, it is sufficient to show that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are unitary equivalent each other.

Clearly, $\mathfrak{B}_1$ is algebraically isomorphic to $\mathfrak{B}_2$. Denote elements of $\mathfrak{B}_1$ by $A^{(1)}, B^{(1)},\dots$ and corresponding elements of $\mathfrak{B}_2$ by $A^{(2)}, B^{(2)}, \dots$. If we put $P^1y_1=z_1$, then $[\mathfrak{B}_1'z_1]=\mathfrak{M}_1$ is obvious. Accordingly, the state τ_1 on $\mathfrak{B}_1$ which is defined by

$$\tau_1(A^{(1)})=(A^{(1)}y_1, y_1),$$

is considered as the restriction of σ_1 on $P^1\mathfrak{A}_1P^1$. By an analogous way to Lemma 1, there exists a partially isometric operator V in $\mathfrak{A}'$ such that

$$[P^1\mathfrak{A}_1P^1x_1]=[P^1\mathfrak{A}_1'P^1x_1]=\mathfrak{M}_1.$$

Furthermore, for any $A^1\in\mathfrak{A}_1$,

$$\sigma_1(P^1A^1P^1)=(A^1z_1, z_1)=(A^1V^*x_1, V^*x_1)=(A^1x_1, x_1)=(P^1A^1P^1x_1,x_1).$$

This shows that

$$\tau_1(A^{(1)})=(A^{(1)}x_1, x_1).$$

By the definitions of P^2 and y_2, we can prove that $x_2\in\mathfrak{M}_2$ and $[\mathfrak{A}_2'y_2]$ is the range of P^2. In other words, $[\mathfrak{B}_2'y_2]=\mathfrak{M}_2$. By an analogous way to the preceding, there exists a vector x_2 in $\mathfrak{M}_2$ such that $[\mathfrak{B}_2x_2]=[\mathfrak{B}_2'x_2]=\mathfrak{M}_2$ and the restriction of σ_2 on $\mathfrak{M}_2$ is

$$\tau_2(A^{(2)})=(A^{(2)}x_2, x_2).$$

Therefore, we have

$$\tau_1(A^{(1)})=\tau_2(A^{(2)}).$$

Finally, it is obvious that $A^{(1)}x_1\to A^{(2)}x_2$ is an isomorphic mapping from the dense manifold of $\mathfrak{M}_1$ to the one of $\mathfrak{M}_2$. Therefore this mapping is uniquely extended to a unitary mapping U from $\mathfrak{M}_1$ onto $\mathfrak{M}_2$. And

$$(U^*A^{(2)}UB_1^{(1)}x_1, B_2^{(1)}x_2)=(A^{(2)}B_1^{(2)}x_2, B_2^{(2)}x_2)=(A^{(1)}B_1^{(1)}x_1, B_2^{(1)}x_1),$$

that is, $\mathfrak{B}_1$ is unitary equivalent to $\mathfrak{B}_2$. This proves the theorem.

This theorem is stated as following in the terminology of F. J. Murray and J. von Neumann[3].

Corollary. The spatial type of a factor of type III is uniquely determined by the algebraical type of it.

Remark. The theorem is still true for the non-factor case.

The detailed proof will be appeared in elsewhere.

An analogous theorem was announced by E. L. Griffin[4].

References

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4) E. L. Griffin: Isomorphism of rings of type III, Bull. Amer. Math. Soc., **58** 480 (1952).