108. *Structure of a Riemann Space*

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Let V_n be an n-dimensional Riemann space with positive definite line element

$$ds^2 = g_{ij}(x)\,dx^i\,dx^j \quad (i, j = 1, 2, \dots, n)$$

in each of its coordinate neighborhoods. Let Γ^i_{jk} be the Christoffel symbols of the second kind made by g_{ij}.

$$R^i_{jhk} = \frac{\partial \Gamma^i_{jk}}{\partial x^h} - \frac{\partial \Gamma^i_{jh}}{\partial x^k} - \Gamma^m_{jh}\Gamma^i_{mk} + \Gamma^m_{jk}\Gamma^i_{mh},$$

$$R_{jh} = R^k_{jhk}, \quad R = g^{ij} R_{ij}$$

are the components of the Riemann-Christoffel tensor, the Ricci tensor and the scalar curvature of the space.

In a previous paper[1], the author has investigated the spaces whose Ricci tensors satisfy the conditions

(a) $$R^k_i R^j_k = \frac{1}{n-1} R R^j_i,$$

(b) $$R^j_{i,k} = 0$$

where a comma " , " denotes the covariant differentiation of the spaces. The first of these conditions is analogous to the condition for Einstein spaces, i.e.

$$R^j_i = \frac{1}{n} R\,\delta^j_i \quad (\delta^j_i = 1 \text{ or } 0, \text{ as } i = j \text{ or } i \neq j)$$

which is equivalent to the condition

$$R^k_i R^j_k = \frac{1}{n} R R^j_i.$$

In connection with Theorem 4 in the paper above, we shall prove a more precise theorem as follows:

Theorem. *If a Riemann space satisfies the conditions* (a), (b), *then it is an Einstein space with zero scalar curvature or a product space with an Einstein space (a surface of constant curvature) and a straight line.*

Proof. Let us put

(1) $$W_{ij} = R_{ij} - \frac{1}{n-1} R g_{ij}.$$

Then we have

$$W = g^{ij} W_{ij} = R - \frac{n}{n-1} R = -\frac{1}{n-1} R,$$

1) T. Ōtsuki: On Some Riemann Spaces, Math. J. Okayama University, Vol. 3, No. 1, pp. 65–88.

hence

(2) $$R_{ij} = W_{ij} - W g_{ij}.$$

Since we have by (2)

$$R_i^k R_k^j = (W_i^k - W\delta_i^k)(W_k^j - W\delta_k^j) = W_i^k W_k^j - 2W W_i^j + W W \delta_i^j,$$

$$\frac{1}{n-1} R R_i^j = - W W_i^j + W W \delta_i^j,$$

the condition (a) becomes

(a′) $$W_i^k W_k^j = W W_i^j.$$

At any point, if we take a suitable rectangular frame, we may put

$$(W_i^j) = \begin{pmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_n \end{pmatrix}.$$

Then (a′) becomes

$$(w_i)^2 = (\sum_j w_j) w_i, \quad i = 1, 2, \dots, n.$$

If $w_1, \dots, w_m = 0$, $w_{m+1}, \dots, w_n \neq 0$, we have

$$w_{m+1} = \cdots = w_n = \sum_j w_j,$$

hence $m = n-1$. It follows that if $W \neq 0$, there exists a unit vector $v_i(x)$ such that

(3) $$W_{ij} = W v_i v_j.$$

Now, we get from (b) $R_{,i} = 0$, i.e. R is a constant, hence W is also a constant. Accordingly (b) may be replaced by the condition

(b′) $$W_{ij,k} = 0.$$

We get from (3), (b′)

$$v_{i,k} v_j + v_i v_{j,k} = 0.$$

On the other hand, since v_i is a unit vector, we have

$$v_{i,k} v^i = 0.$$

Hence we get from the two relations above

(4) $$v_{i,j} = 0.$$

Therefore, there exists a scalar $y(x)$ such that

(5) $$v_i = \partial y / \partial x^i.$$

Making use of orthogonal trajectories of the family of the hypersurfaces on which $y(x)$ is constant, we can choose a coordinate system $x^1, \dots, x^{n-1}$, y such that

$$g_{\lambda n}(x, y) = 0, \quad \lambda = 1, 2, \dots, n-1.$$

Then by (5) we have $(v_i) = (0, 0, \dots, 1)$ in the coordinate system, it follows that $1 = g^{ij} v_i v_j = g^{nn}(x, y)$ and $g_{nn}(x, y) = 1$.

Furthermore, we have

$$v_{i,j} = - \Gamma_{ij}^k v_k = - \Gamma_{ij}^n = 0,$$

hence

$$\Gamma_{\lambda\mu}^n = \Gamma_{\lambda n \mu} = - \frac{1}{2} \frac{\partial g_{\lambda\mu}}{\partial y} = 0, \quad \lambda, \mu = 1, 2, \dots, n-1.$$

Accordingly $g_{\lambda\mu}(x, y)$ is independent on y, the line element of the space is of the form

$$ds^2 = g_{\lambda\mu}(x)\, dx^\lambda\, dx^\mu + dy\, dy\,,$$ [2]

which shows that the space is a product of an $(n-1)$-dimensional Riemann space V_{n-1} with line element $ds^2 = g_{\lambda\mu}(x)\, dx^\lambda\, dx^\mu$ and a straight line.

On the other hand, since $\Gamma^n_{ij} = \Gamma^i_{nj} = 0$, we have

$$R_{jn} = R^{\mu}_{jn\mu} = \frac{\partial \Gamma^{\mu}_{j\mu}}{\partial y} - \frac{\partial \Gamma^{\mu}_{jn}}{\partial x^\mu} - \Gamma^i_{jn}\Gamma^{\mu}_{i\mu} + \Gamma^i_{j\mu}\Gamma^{\mu}_{in} = 0\,,$$

hence we get easily

$$\bar{R}_{\lambda\mu\nu\sigma} = R_{\lambda\mu\nu\sigma}\,,\quad \bar{R}_{\lambda\mu} = R_{\lambda\mu}\,,\quad \bar{R} = R$$

where $\bar{R}_{\lambda\mu\nu\sigma}$, $\bar{R}_{\lambda\mu}$ *and* $\bar{R}$ are the components of the Riemann-Christoffel tensor, the Ricci tensor and the scalar curvature of the space V_{n-1}. Accordingly, in the coordinates $x^1, \ldots, x^{n-1}, y$, we have

$$(R^j_i) = \begin{pmatrix} \bar{R}^{\mu}_{\lambda} & 0 \\ 0 & 0 \end{pmatrix}.$$

(a) becomes

$$\bar{R}^{\mu}_{\lambda}\,\bar{R}^{\nu}_{\mu} = \frac{1}{n-1}\,\bar{R}\,\bar{R}^{\nu}_{\lambda}\,.$$

It follows that $\bar{R}^{\mu}_{\lambda} = \dfrac{1}{n-1}\,\bar{R}\,\delta^{\mu}_{\lambda}$, i.e. V_{n-1} is an Einstein space with the same constant scalar curvature as V_n.

Finally, if $W=0$, we get easily $W^j_i=0$, hence by (2) $R_{ij}=0$. V_n is an Einstein space with zero scalar curvature. The proof is complete.

2) In the following, we assume that $\lambda, \mu, \nu, \sigma$ take the values $1, 2, \ldots, n-1$.