122. A Necessary Unitary Field Theory as a Non-Holonomic Parabolic Lie Geometry Realized in the Three-Dimensional Cartesian Space

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The geometry based upon is the author's non-holonomic parabolic Lie geometry[3) *)], which is situated among other branches of geometry as follows: (Euclidean geometry): (Non-Euclidean geometry) = (parabolic Lie geometry): (Lie geometry) = (non-holonomic parabolic Lie geometry): (non-holonomic Lie geometry). Instead of the quadratic differential form:

(0.1) $$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{\underline{\mu\nu}}dx^\mu dx^\nu + g_{\underset{\vee}{\mu\nu}}dx^\mu dx^\nu\,,$$

we take the linear vector form

(0.2) $$\gamma_5\omega^5 = \gamma_l\omega^l\,,\ (\omega^l = \omega^l_\mu dx^\mu\,,\ l = 1, 2, 3, 4)\,,$$

such that

(0.3) $$dsds = \omega^5\omega^5 = \omega^l\omega^l\,,$$

where in Einstein's notation[1)] we have

(0.4) $$g_{\underline{\mu\nu}} = \omega^l_\mu\omega^l_\nu\,,$$

(0.5) $$g_{\underset{\vee}{\mu\nu}} = \gamma_4\gamma_1(\omega^4_\mu\omega^1_\nu - \omega^4_\nu\omega^1_\mu) + \cdots + \gamma_2\gamma_3(\omega^2_\mu\omega^3_\nu - \omega^3_\mu\omega^2_\nu)\cdots +\,,$$

and

(0.6) $$\gamma_1^2=\gamma_2^2=\gamma_3^2=-\gamma_4^2=\gamma_5^2=1\,,\ \gamma_4=i\gamma_5\,,\ \gamma_2\gamma_3+\gamma_3\gamma_2=0\,,\ \text{etc.},$$
$$\gamma_4\gamma_1+\gamma_1\gamma_4=0\,,\ \text{etc.},\ \gamma_5\gamma_1+\gamma_1\gamma_5=0\,,\ \text{etc.},$$

the $\gamma_1, \gamma_2, \gamma_3, \gamma_5$ being the Pauli's 4-4-matrices. Starting from (0.2) and pursuing necessities stepwise, the author will develop a unitary field theory.

1. *Realization of the Non-Holonomic Parabolic Lie Geometry in the Cartesian Space.* The said geometry will be realized in the three-dimensional Cartesian space provided with the Cartesian coordinates (ξ^i), $(i=1, 2, 3)$, such that

(1.1) $$d\xi^l = \omega^l\,,$$

(1.2) $$d\xi^4 = \omega^4 = dr\,,$$

the r being the radius of the oriented sphere with center $P(\xi^i)$. We adopt a double use for ds:

a vector (0.2) with components ω^l.	the common tangential segment $ds=idS$ of the oriented sphere (P, r) with its consecutive one.

The quantity $ds=idS$ is purely imaginary, when

*) The ciphers in the square brackets refer to the References attached to the end of this paper.

$$d\xi^i d\xi^i - dr^2 = d\sigma^2 - dr^2 < 0\,.$$

If we put

(1.3) $$u^l = \frac{\omega^l}{d\sigma}, \quad u^5 = \frac{\omega^5}{d\sigma}, \quad (d\sigma^2 = \omega^i\omega^i)\,,$$

the condition (0.3) may be rewritten:

(1.4) $$u^A u^A = 0\,, \qquad (A = 1, 2, \dots, 5)\,.$$

2. *Problem (Two Particles Problem).* We consider two particles O and P respectively charged with rest-masses $\overline{m}_0$, m_0 and with constant electricity $-\bar{e}$, $-e$, which make motions relative to each other. Then both O and P emit gravitational energy and electric energy spherically. The law of motion is required. In Art. 4, this problem will be solved.

3. *General-Relativistically Generalized Maxwell's Equations.* Introducing the notations: ϕ^i=electromagnetic vector potential, $(i=1,2,3)$; $-\phi^4$=electrostatic potential; σ^i=current components; σ^4=electric density, $\Phi=\gamma_l\phi^l$, $J=-\gamma_i\sigma^i+\gamma_4\sigma^4$, X^i=electric intensity, α^i=magnetic intensity, the author has proved[2] that the eight components of the single equation

(3.1) $$4\frac{\partial^2\Phi}{\omega^5\omega^5} = J$$

are the general-relativistically generalized Maxwell's equations:

(3.2) $$\begin{cases} \dfrac{\partial X^i}{\omega^i} = \sigma^4\,, & -\dfrac{\partial X^i}{\omega^4} - \left(\dfrac{\partial \alpha^j}{\omega^k} - \dfrac{\partial \alpha^k}{\omega^j}\right) = \sigma^i\,, \\ \dfrac{\partial \alpha^i}{\omega^i} = 0\,, & \dfrac{\partial \alpha^i}{\omega^4} - \left(\dfrac{\partial X^j}{\omega^k} - \dfrac{\partial X^k}{\omega^j}\right) = 0\,. \end{cases}$$

4. *Solution of the Problem Stated in Art.* 2. Take a Cartesian system (ξ^i) with the position of the first particle O as origin. Then we can put[2]:

(4.1) $$d\xi^l = \omega^l\,, \qquad d\xi^4 = \omega^4 = dr\,,$$

where r is the radius of the oriented sphere with center (ξ^i), which is the energy level emitted from the particle $P(\xi^i)$. In case $d\sigma^2 - dr^2 < 0$, the sphere (P, r) encloses the particle O, which emits gravitational energy due to $\overline{m}_0$ and electric energy due to $-\bar{e}$ spherically, the energy level being the sphere (O, S) with center O and radius S. Put

(4.2) $$E = \frac{dr}{dt} = \text{radial energy emitted from } P$$
$$= \text{radial velocity of the energy level } (P, r)\,,$$

(4.3) $$\bar{E} = \frac{dS}{dt} = \text{radial energy emitted from } O$$
$$= \text{radial velocity of the energy level } (O, S)\,.$$

Let $\bar{\varphi}^i$ = electromagnetic vector potential for O,

(4.4) $e\phi^5 = m_0$ (gravitational static potential for P),

(4.5) $\bar{e}\bar{\phi}^4 = \overline{m}_0$ (gravitational static potential for O),

(4.6) p^i = momentum components for P,
(4.7) $\bar{p}^i$ = momentum components for O,
(4.8) Ep^4 = total energy for P in case of no gravitation,
(4.9) $\bar{E}p^5$ = total energy for O in case of no gravitation,
(4.10) Ep^5 = total energy of P for the case of no electric field
= E times the corresponding momentum,
(4.11) $\bar{E}\bar{p}^4$ = total energy for O in case of no electric field
= $\bar{E}$ times the corresponding momentum.

Then

$$(4.12)\qquad (Ep^i+e\phi^i+\bar{E}\bar{p}^i+\bar{e}\bar{\varphi}^i)=\left(mE^2\frac{d\sigma}{dr}+\bar{m}\bar{E}^2\frac{d\sigma}{dS}\right)u^i,$$

$$(4.13)\qquad (Ep^4+e\phi^4+\bar{E}\bar{p}^4+\bar{e}\bar{\varphi}^4)=\left(mE^2\frac{d\sigma}{dr}+\bar{m}\bar{E}^2\frac{d\sigma}{dS}\right)u^4,$$

$$(4.14)\qquad (Ep^5+e\phi^5+\bar{E}\bar{p}^5+\bar{e}\bar{\varphi}^5)=\left(mE^2\frac{d\sigma}{dr}+\bar{m}\bar{E}^2\frac{d\sigma}{dS}\right)u^5,$$

where $m=m_0\dfrac{dr}{dS}$ and $\bar{m}=\bar{m}_0\dfrac{dS}{dr}$ are longitudinal masses. (4.12), (4.13), (4.14) and (0.2) with $\omega^5=\omega^5_\mu(x^\lambda)dx^\mu$ give

$$(4.15)\qquad \gamma_l(Ep^l+e\phi^l+\bar{E}\bar{p}^l+\bar{e}\bar{\varphi}^l)=\gamma_5(Ep^5+e\phi^5+\bar{E}\bar{p}^5+\bar{e}\bar{\varphi}^5).$$

For $\gamma_l\phi^l-\gamma_5\phi^5=\Psi$, $\gamma_lp^l-\gamma_5p^5=P$, etc., (4.15) becomes

$$(4.16)\qquad Ep+e\Psi+\bar{E}\bar{p}+\bar{e}\bar{\Psi}=0.$$

Applying the operator

$$(4.17)\qquad 2\gamma_5\frac{\partial}{\omega^5}=\gamma_l\frac{\partial}{\omega^l}=\gamma_l\frac{\partial}{\partial\xi^l}$$

to (4.16), we have

$$(4.18)\qquad 2\gamma_5\frac{\partial}{\omega^5}(EP+e\Psi+\bar{E}\bar{P}+\bar{e}\bar{\Psi})=\frac{\partial}{\omega^l}(Ep^l+e\phi^l+\bar{E}\bar{p}^l+\bar{e}\bar{\varphi}^l)$$
$$-\gamma_4\gamma_i(\mathfrak{X}^i+eX^i+\bar{\mathfrak{X}}^i+\bar{e}\bar{X}^i)+\gamma_j\gamma_k(a^i+e\alpha^i+\bar{a}^i+\bar{e}\bar{\alpha}^i)$$
$$-2\frac{\partial}{\omega^5}(Ep^5+e\phi^5+\bar{E}\bar{p}^5+\bar{e}\bar{\varphi}^5)=0,$$

where

$$(4.19)\qquad \mathfrak{X}^i=\frac{\partial(Ep^4)}{\omega^i}+\frac{\partial(Ep^i)}{\omega^4},\ \text{etc.},$$

$$(4.20)\qquad a^i=\frac{\partial(Ep^k)}{\omega^j}-\frac{\partial(Ep^j)}{\omega^k},\ \text{etc.},$$

$$(4.21)\qquad X^i=\frac{\partial\phi^4}{\omega^i}+\frac{\partial\phi^i}{\omega^4},\ \text{etc.},$$

$$(4.22)\qquad \alpha^i=\frac{\partial\phi^k}{\omega^j}-\frac{\partial\phi^j}{\omega^k},\ \text{etc.}$$

Introducing the continuity condition

$$(4.23)\qquad \frac{\partial}{\omega^l}(Ep^l+e\phi^l+\bar{E}\bar{p}^l+\bar{e}\bar{\phi}^l)-2\frac{\partial}{\omega^5}(Ep^5+e\phi^5+\bar{e}\bar{p}^5+e\bar{\varphi}^5)=0$$

and applying (4.17) once more, we obtain the generalization of the Maxwell's equations:

$$(4.24)\qquad \frac{\partial}{\omega^i}(\mathfrak{X}^i+eX^i+\overline{\mathfrak{X}}^i+\bar{e}\overline{X}^i)=\varepsilon^4+\sigma^4+\bar{\varepsilon}^4+\bar{\sigma}^4,$$

$$(4.25)\qquad \frac{\partial}{\omega^i}(a^i+e\alpha^i+\bar{a}^i+\bar{e}\bar{\alpha}^i)+\frac{\partial}{\omega^j}(\mathfrak{X}^k+eX^k+\overline{\mathfrak{X}}^k+\bar{e}\overline{X}^k)-\frac{\partial}{\omega^k}(\mathfrak{X}^j+e\mathfrak{X}^j+\overline{\mathfrak{X}}^j+\bar{e}\overline{X}^j)=0,$$

$$(4.26)\qquad \frac{\partial}{\omega^j}(a^k+e\alpha^k+\bar{a}^k+\bar{e}\bar{\alpha}^k)-\frac{\partial}{\omega^k}(a^j+e\alpha^j+\bar{a}^j+\bar{e}\bar{\alpha}^j)-\frac{\partial}{\omega^4}(\mathfrak{X}^i+eX^i+\overline{\mathfrak{X}}^i+\bar{e}\overline{X}^i)=\varepsilon^i+\sigma^i+\bar{\varepsilon}^i+\bar{\sigma}^i,$$

$$(4.27)\qquad \frac{\partial}{\omega^i}(a^i+e\alpha^i+\bar{a}^i+\bar{e}\bar{\alpha}^i)=0,$$

where ε^4=gravitational density due to P, $\bar{\varepsilon}^4$=gravitational density due to O, ε^i=components of "gravitational current" due to P, $\bar{\varepsilon}^i$ =those due to O. Perhaps ε^i, $\bar{\varepsilon}^i$, ε^4 and $\bar{\varepsilon}^4$ will be very small compared with σ^i, $\bar{\sigma}^i$, σ^4 and $\bar{\sigma}^4$ respectively.

5. *Generalized Dirac Equations.* Put

$$(5.1)\qquad \psi=2\gamma_5\frac{\partial}{\omega^5}(EP+e\Psi+\overline{E}\overline{P}+\bar{e}\overline{\Psi})=-\gamma_4\gamma_i(\mathfrak{X}^i+eX^i+\overline{\mathfrak{X}}^i+\bar{e}\overline{X}^i)+\gamma_j\gamma_k(a^i+e\alpha^i+\bar{a}^i+\bar{e}\bar{\alpha}^i),$$

and applying (4.17) once more, we obtain

$$(5.2)\qquad 4\frac{\partial^2}{\omega^4\omega^4}(EP+e\Psi+\overline{E}\overline{P}+\bar{e}\overline{\Psi})\equiv 2\gamma_5\frac{\partial\psi}{\omega^5}\equiv\gamma_l\frac{\partial\psi}{\omega^l}=0,$$

which leads us to the generalized Dirac equation:

$$(5.3)\qquad \left[\gamma_l\left(\frac{h}{2\pi i}\frac{\partial}{\omega^l}+e\phi^l+\frac{h}{2\pi i}\overline{E}\frac{\partial}{\omega^l}+\bar{e}\bar{\phi}^l\right)+\gamma_5(m_0E+\overline{m}_0\overline{E})\right]\psi=0$$

by a process similar to the usual one.

Applying (4.17) once more, we obtain

$$(5.4)\qquad 8\frac{\partial^3}{\omega^5\omega^5\omega^5}(EP+e\Psi+\overline{E}\overline{P}+\bar{e}\overline{\Psi})\equiv 4\frac{\partial^2}{\omega^5\omega^5}\psi=\gamma_k\frac{\partial}{\omega^k}\gamma_l\frac{\partial}{\omega^l}\psi=0,$$

which leads us to a generalized Schrödinger equation.

References

1) Einstein, A.: The Meaning of the Relativity. Fourth Edition Appendix 2 (1953).

2) Takasu, T.: The General Relativity as a Three-Dimensional Non-Holonomic Laguerre Geometry, Its Gravitation Theory and Its Quantum Mechanics. The Yokohama Math. Jour., **1**, 89-104 (1953).

3) Takasu, T.: A Combined Field Theory as a Three-Dimensional Non-Holonomic Parabolic Lie Geometry and Its Quantum Mechanics. The Yokohama Math. Jour., **1**, 105-116 (1953).