13. On Rings of Continuous Functions and the Dimension of Metric Spaces

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M. Katětov [1] has once established an interesting theory on a relation between the inductive (Menger-Urysohn) dimension of a compact space R and the structure of the ring of all continuous functions on R. The purpose of this brief note is to give a slight extension to Katětov's theory for a metric space while simplifying his discussion.

According to [1], we consider an *analytical ring*, i.e. a commutative topological ring with a unit e and a continuous real scalar multiplication. A subring C_1 of an analytical ring C is called *analytically closed* if

(1) $\lambda e \in C_1$ for any real λ, (2) $x \in C_1$ whenever $x \in C$, $x^n + a_1 x^{n-1} + \cdots + a_n = 0$, $a_i \in C_1$, (3) $\bar{C}_1 = C_1$.

Let C' be a subset of C; then a subset M of C is called an analytical base of C' in C if there exists no analytically closed subring $C_1 \not\supset C'$ containing M. The least number of an analytical base of C' in C is called the analytical dimension of C' in C and denoted by $\dim(C', C)$. The ring $C(R)$ of all bounded real-valued continuous functions of R is an analytical ring as for its strong topology. We denote by $U(R)$ the subset of $C(R)$ consisting of all uniformly continuous functions. Furthermore, according to [2], we call a continuous mapping f of a metric space R into a metric space S *uniformly* 0-*dimensional* if for any $\varepsilon > 0$ there exists $\eta > 0$ such that $\delta(U) < \varepsilon$ whenever $U \subset R$, diam $f(U) < \eta$, where $\delta(U) < \varepsilon$ means the fact that there exists an open covering $\mathfrak{V}$ of U such that mesh $\mathfrak{V} = \sup\{\text{diam } V \mid V \in \mathfrak{V}\} < \varepsilon$ and order $\mathfrak{V} \leqq 1$. The covering dimension of R or the strong inductive dimension of R as the same is denoted by $\dim R$. Now we can prove the following

Theorem. $\dim R = \dim(U(R), C(R))$ *for every locally compact, metric space* R.

To establish this theorem we prove some lemmas.

Lemma 1. *Let* $f(x) = (f_1(x), \cdots, f_n(x))$ *be a uniformly* 0-*dimensional, bounded mapping of a metric space* R *into the n-dimensional Euclidean space* E_n. *Let* C_1 *be an analytically closed subring of* $C(R)$ *containing* $f_1, \cdots f_n$; *then for every sets* F *and* G *of* R *with distance* $(F, G) = d(F, G) > 0$, *there exists* $g \in C_1$ *such that* $g(F) \geqq 1$, $g(G) = 0$, *where* $g(F) \geqq 1$, *for example, means that* $g(x) \geqq 1$ *for every* $x \in F$.

Proof. Let $d(F,G)=\varepsilon>0$ and take $\eta>0$ such that diam $f(U)<\eta$ for $U\subset R$ implies $\delta(U)<\varepsilon$. Choosing $\xi>0$ such that diam $\prod_{i=1}^{n}(r_i-2\xi, r_i+2\xi)<\eta$ for every r_i, we cover $f(R)$ with finitely many cubes

$$I_k=\prod_{i=1}^{n}[r_{ki}-\xi,\ r_{ki}+\xi],\quad k=1,\cdots,l.$$

Let

$$U_k=f^{-1}(I_k),\ V_k=f^{-1}(J_k),$$

where

$$J_k=\prod_{i=1}^{n}(r_{ki}-2\xi,\ r_{ki}+2\xi).$$

It easily follows from $f_i\in C_1$ that $f_{ki}=(2\xi-|f_i-r_{ki}|)\frac{1}{\xi}\in C_1$, and hence $\bar{f}_k=\prod_{i=1}^{n}f_{ki}\in C_1$. Then

$$\bar{f}_k(x)\geqq 1 \quad \text{for every} \quad x\in U_k,$$
$$\bar{f}_k(x)>0 \quad \text{for every} \quad x\in V_k,$$
$$\bar{f}_k(x)=0 \quad \text{for every} \quad x\in \bar{V}_k-V_k.$$

Since diam $f(V_k)=$diam $J_k<\eta$, we can find an open covering $\mathfrak{V}_k$ of V_k with mesh $\mathfrak{V}_k<\varepsilon$, order $\mathfrak{V}_k\leqq 1$. It is easy to see that $S(F,\mathfrak{V}_k)=W_k$ is an open closed set of V_k satisfying $W_k\frown G=\phi$, $W_k\supset F\frown V_k$. Now we define a function g_k by

$$g_k(x)=\bar{f}_k(x),\quad x\in W_k,$$
$$g_k(x)=0,\quad x\notin W_k.$$

Then since g_k clearly satisfies $g_k\in C(R)$ and $g_k^2-\bar{f}_kg_k=0$ for $\bar{f}_k\in C_1$, we get $g_k\in C_1$ satisfying $g_k(F\frown U_k)\geqq 1$, $g_k(G)=0$, $g_k\geqq 0$. Letting $g=\sum_{k=1}^{l}g_k$ we have an element g of C_1 satisfying $g(F)\geqq 1$, $g(G)=0$, $g\geqq 0$.

Lemma 2. $\dim R\geqq \dim(U(R),C(R))$ *for every metric space R.*

Proof. If $\dim R\leqq n$, then by [2] there exists a uniformly 0-dimensional, bounded mapping $f(x)=(f_1(x),\cdots,f_n(x))$ of R into E_n. Hence any analytically closed subring C_1 of $C(R)$ containing $f_1,\cdots,f_n$ also contains, for every disjoint closed sets F and G with $d(F,G)>0$, $\varphi\in C_1$ such that $\varphi(F)=0$, $\varphi(G)\geqq 1$ by Lemma 2. Hence by an analogous theorem to that of E. Hewitt [3, Theorem 1], we get, for every $\bar{\varphi}\in U(R)$ and $\varepsilon>0$ a polynomial $P(\varphi_1,\cdots,\varphi_k)$ in $\varphi_i\in C_1$, $i=1,\cdots,k$ such that $|\bar{\varphi}-P(\varphi_1,\cdots,\varphi_n)|<\varepsilon$. Therefore $\bar{\varphi}\in\bar{C}_1=C_1$, which implies $C_1\supseteqq U(R)$. Thus $(f_1,\cdots,f_n)$ is an analytical base of $U(R)$ in $C(R)$, i.e. $\dim(U(R), C(R))\leqq n$.

Lemma 3. $\dim R\leqq \dim(U(R),C(R))$ *for every locally compact, metric space R.*

Proof. Let $(f_1,\cdots,f_n)$ be an analytical base of $U(R)$ in $C(R)$; then $f(x)=(f_1(x),\cdots,f_n(x))$ is a bounded continuous mapping of R onto a subset $f(R)$ of E_n. Since R is locally compact, there is a locally finite closed covering $\{R_\alpha\mid\alpha\in\Omega\}$ consisting of compact sets R_α. Let $\mathfrak{U}$ be any finite open covering of R_α; then there exists, for every $q\in f(R)$, a nbd (=neighborhood) $V(q)$ of q in $f(R)$ such that $\delta(f^{-1}V(q))\leqq\mathfrak{U}$, i.e. there exists an open covering $\mathfrak{B}$ of $f^{-1}V(q)$ satisfying $\mathfrak{B}<\mathfrak{U}$ in R_α and order $\mathfrak{B}\leqq 1$. It is enough to prove this proposition just for every binary open covering $\mathfrak{U}$ of R_α. For we can find, for every finite open covering $\mathfrak{U}$ of R_α, binary open coverings $\mathfrak{U}_1,\cdots,\mathfrak{U}_k$ of R_α satisfying $\mathfrak{U}_1\wedge\cdots\wedge\mathfrak{U}_k<\mathfrak{U}$. Then $\delta(f^{-1}V_i(q))\leqq\mathfrak{U}_i$, $i=1,\cdots,k$ for nbds $V_i(q)$, $i=1,\cdots,k$ of q imply $\delta(f^{-1}\bigcap_{i=1}^{k}V_i(q))\leqq\mathfrak{U}$. Now assume the contrary, i.e. let F and G be disjoint closed sets of R_α such that $\delta(f^{-1}V(q))\not\leqq\{F^c, G^c\}$ for every nbd $V(q)$ of q.

Let $D=\{g\mid g\in C(R)$, for every $\varepsilon>0$, there exist a nbd $V(q)$ of q in $f(R)$ and an open covering $\mathfrak{U}$ of $f^{-1}V(q)$ such that mesh $g(\mathfrak{U})<\varepsilon$ and order $\mathfrak{U}\leqq 1\}$, where $g(\mathfrak{U})$ denotes the covering $\{g(U)\mid U\in\mathfrak{U}\}$ then D is an analytically closed subring containing $f_1,\cdots,f_n$. Let us just show that $g\in D$ whenever $g\in C(R)$, $g^n+a_1g^{n-1}+\cdots+a_n=0$, $a_i\in D$, where this n is not related with the number of f_i. Let us denote by $g_k(b_1,\cdots,b_n)$, $k=1,2,\cdots,n$ the n roots of the equation

$$y^n+b_1y^{n-1}+\cdots+b_n=0.$$

Let $|a_i|\leqq K$, $i=1,\cdots,n$; then since $g_k(b_1,\cdots,b_n)$ are continuous functions of $b_1,\cdots,b_n$ and accordingly are uniformly continuous for $|b_i|\leqq K$, $i=1,\cdots,n$, for any $\varepsilon>0$ we can find $\delta>0$ such that

$$|b_i-b'_i|<\delta,\ |b_i|\leqq K,\ |b'_i|\leqq K,\ i=1,\cdots,n \text{ imply}$$

$$|g_k(b_1,\cdots,b_n)-g_k(b'_1,\cdots,b'_n)|<\frac{\varepsilon}{n},\ k=1,\cdots,n.$$

Now let $V(q)$ be a nbd of q and $\mathfrak{U}=\{U_\gamma\mid\gamma\in\Gamma\}$ an open covering of $f^{-1}V(q)$ such that mesh $a_i(\mathfrak{U})<\delta$, $i=1,\cdots,n$ and order $\mathfrak{U}\leqq 1$. Moreover, let

$$\{x\mid g_k(a_1(x),\cdots,a_n(x))-g(x)=0,\ x\in U_\gamma\}=U_{k\gamma},$$
$$\{U_{k\gamma}\mid k=1,\cdots,n\}=\mathfrak{U}_\gamma,$$
$$\{S^n(U_{k\gamma},\ \mathfrak{U}_\gamma)\mid U_{k\gamma}\in\mathfrak{U}_\gamma\}=\mathfrak{B}_\gamma.$$

Then $\mathfrak{B}_\gamma$ is an open covering of U_γ with order $\mathfrak{B}_\gamma\leqq 1$ and mesh $g(\mathfrak{B}_\gamma)<\varepsilon$; hence $\mathfrak{B}=\smile\{\mathfrak{B}_\gamma\mid\gamma\in\Gamma\}$ is an open covering of $f^{-1}V(q)$ with order $\mathfrak{B}\leqq 1$ and mesh $g(\mathfrak{B})<\varepsilon$. Thus we get $g\in D$. Since R_α is compact, it must be $d(F,G)>0$, and hence there exists a function $h\in U(R)$ such that $h(F)=0$, $h(G)=1$. However, from the assumption D does not contain such a function h, which contradicts the fact that $(f_1,\cdots,f_n)$ is an analytical base of $U(R)$ in $C(R)$. Hence for every finite open covering $\mathfrak{U}$ of R_α and for every point q of $f(R)$ there exists a nbd $V(q)$ of q satisfying $\delta(f^{-1}V(q))\leqq\mathfrak{U}$. Take an open refinement $\mathfrak{B}=\{V_\gamma\mid\gamma\in\Gamma\}$ of $\{V(q)\mid q\in f(R)\}$ with order $\mathfrak{B}\leqq n+1$. Then since $\delta(f^{-1}(V_\gamma))\leqq\mathfrak{B}$, we can find an open covering $\mathfrak{W}_\gamma$ of $f^{-1}(V_\gamma)$ satisfying $\mathfrak{W}_\gamma<\mathfrak{U}$, order $\mathfrak{W}_\gamma\leqq 1$. Now $\mathfrak{W}=\smile\{\mathfrak{W}_\gamma\mid\gamma\in\Gamma\}$ restricted in R_α is an open refinement of $\mathfrak{U}$ with order $\mathfrak{W}\leqq n+1$. Therefore we can conclude dim $R_\alpha\leqq n$. Hence, by use of the sum-theorem, we get $\dim R \leqq n$.

Combining Lemma 3 with Lemma 2, we can conclude the validity of the theorem.

Incidentally, let us show the following

Corollary. *$U(R)$ of every metric space R has an analytical base in $C(R)$ consisting of countably many elements.*

While checking up the proofs of Lemmas 1, 2, we know that this corollary is a direct consequence of the following

Lemma 4. *Every metric space R can be mapped into the Hilbert cube I_w by a uniformly 0-dimensional mapping.*

Proof. Since, by [4], every metric space R can be imbedded in a product of countably many one-dimensional metric spaces, we can conclude this lemma from the fact owing to [2] that every one-dimensional metric space is mapped into E_1 by a uniformly 0-dimensional, bounded function.

References

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