92. On the Equivalence of Excessive Functions and Superharmonic Functions in the Theory of Markov Processes. II

By Takesi WATANABE

Department of Applied Science, Kyushu University, Fukuoka

(Comm. by Z. SUETUNA, M.J.A., July 12, 1962)

1. **Introduction.** Let X be a strict Markov process on a locally compact Hausdorff space.[1] Consider a family $\mathcal{F}_x$ of admissible subsets which depends on the state $x \in S$ and put $\mathcal{F}=\bigcup_{x \in S}\mathcal{F}_x$. A nonnegative and $\mathcal{B}$-measurable function u is called *$\mathcal{F}$-superharmonic* if

(1.1) $$u(x) \geqq H_{\hat{\sigma}_A}u(x) \quad \text{for every } A \in \mathcal{F}_x,$$

and *$\mathcal{F}$-continuous* if

(1.2) $H_{\hat{\sigma}_{A_n}}u(x) \to u(x)$ for any A_n in $\mathcal{F}_x$ such that $P_x\{\hat{\sigma}_{A_n} \downarrow 0\}=1$,

where $\hat{\sigma}_A$ is the positive hitting time for the set A, that is, $\hat{\sigma}_A =\inf\{t>0, x_t \in A\}$.[2] In these terminologies, a nonnegative superharmonic function u in the usual sense is caught in the following way. Let S be an open domain of R^n and X, the Brownian motion on S. Consider the euclidean metric ρ and denote the ball $\{y; \rho(x, y) < r\}$ by $U_{x,r}$. A nonnegative function u is superharmonic in the classical sense if and only if it is $\mathcal{F}$-superharmonic and $\mathcal{F}$-continuous with respect to the family $\mathcal{F}$ induced by $\mathcal{F}_x=\{\overline{U}^c_{x,r};\ r>0 \text{ and } \overline{U}_{x,r} \text{ is compact}\}$.[3]

According to Proposition 2.4 of (I), any excessive function is $\mathcal{F}$-superharmonic and $\mathcal{F}$-continuous for any family $\mathcal{F}$. The converse problem is now stated as follows: *For what family $\mathcal{F}$ does it hold that every $\mathcal{F}$-superharmonic and $\mathcal{F}$-continuous function is excessive?* This problem is solved for a sufficiently large $\mathcal{F}$, affirmatively. For example, a theorem due to Dynkin [1][4] asserts that it is enough to take the family $\mathcal{F}$ such that $\mathcal{F}_x$={every compact sets in S}, if X satisfies the quasi-continuity from the left.[5] But this theorem seems

1) In this paper we shall use the terminologies and notations of the previous paper [4] with no special reference. In the following, it will be quoted as (I).

2) Since A is admissible, the nonnegative hitting time $\sigma_A=\inf\{t \geqq 0, x_t \in A\}$ is a Markov time. Noting that $t+\sigma_A(w_t^+) \downarrow \hat{\sigma}_A(w)$, $\hat{\sigma}_A$ is also a Markov time.

3) In this case, it is shown that the $\mathcal{F}$-continuity with respect to our family is equivalent to the lower semicontinuity, using special properties of the Brownian motion.

4) He stated this theorem without proof. Recently, Motoo has proved it (private communication).

5) For the definition, see [2]. It is known that any Borel set (and therefore any compact set) of S is admissible if X is quasi-continuous from the left.

to be sometimes inconvenient for application. In fact, the family $\mathcal{F}$ in Dynkin's theorem is so large and complicated that it is not easy to verify whether a given function is superharmonic in Dynkin's sense even for the case of a Brownian motion. As was introduced in (I), the function u is called simply *superharmonic* if there exists some open base $\mathcal{U}$ such that u is $\mathcal{F}$-superharmonic with respect to the family $\mathcal{F}$ induced by $\mathcal{F}_x=\{\overline{U}^c;\ U$ is any set of $\mathcal{U}$ containing $x\}$.[6] The purpose of this paper is to prove the following

THEOREM 1. *Let X be a strict Markov process whose Green operator G_α $(\alpha>0)$ maps the family $\boldsymbol{C}(S)$ of all bounded continuous functions into itself. Then every lower semicontinuous superharmonic function is excessive.*

It should be noted that our theorem is still unsatisfactory, because in it we assume the conditions on G_α or the lower semicontinuity of functions which are not essential in probability theory. It is an open problem to the author whether we can always replace the lower semicontinuity by the $\mathcal{F}$-continuity with respect to our family.

2. Generator. In this section we shall summarize some results on the generator due to K. Itô [3].

Let X be a strict Markov process and $\boldsymbol{D}_X(S)$, the set of all bounded $\mathcal{B}$-measurable functions satisfying that, for any x and for any sequence of constant times such that $t_j\downarrow 0$, $f(x_{t_j})\to f(x)$ with P_x-probability 1. In general, $\boldsymbol{D}_X(S)\supseteqq \boldsymbol{C}(S)$. Moreover note that $\boldsymbol{D}_X(S)=\boldsymbol{B}(S)$[7] if X is a regular step process. G_α $(\alpha>0)$ maps $\boldsymbol{D}_X(S)$ into itself and its range $G_\alpha(\boldsymbol{D}_X(S))$ is independent of $\alpha>0$. Denote $G_\alpha(\boldsymbol{D}_X(S))$ by $\mathcal{D}(\mathcal{G})$. G_α $(\alpha>0)$ defines an one-to-one mapping from $\boldsymbol{D}_X(S)$ onto $\mathcal{D}(\mathcal{G})$ and $\mathcal{G}=\alpha-G_\alpha^{-1}(\mathcal{D}(\mathcal{G})\to\boldsymbol{D}_X(S))$ is known to be an operator independent of α. $\mathcal{G}$ is called the generator of X. In particular, if G_0 is also a bounded operator (i.e. $\sup_{x\in S} G_0(x, S)<+\infty$), we have $G_0(\boldsymbol{D}_X(S))=\mathcal{D}(\mathcal{G})$. If A is admissible and $0<E_x(\sigma_A)<+\infty$, we have

$$\left|\mathcal{G}u(x)-\frac{H_A u(x)-u(x)}{E_x(\sigma_A)}\right|\leqq \sup_{y\in A^c}|\mathcal{G}u(x)-\mathcal{G}u(y)| \tag{2.1}$$

for every $u\in\mathcal{D}(\mathcal{G})$. Therefore, if X is a regular step process, putting $A=\{x\}^c$, it follows that

$$\mathcal{G}u(x)=\mathcal{G}_D u(x)\quad \text{for every } u\in\mathcal{D}(\mathcal{G}), \tag{2.2}$$

where $\mathcal{G}_D$ is the Dynkin generator (see (I)).

3. Approximation of a general process by regular step processes. We shall start with the following simple

6) Since $\overline{U}^c$ is open, we have $\hat{\sigma}_{\overline{U}^c}(w)=\sigma_{\overline{U}^c}(w)$ for every w. Therefore, our definition of superharmonic functions is the same as in (I).

7) $\boldsymbol{B}(S)$=the set of all bounded $\mathcal{B}$-measurable functions.

LEMMA 3.1. *Let the strict Markov processes X, $X^{(n)}$ satisfy the conditions that* (1) $\mathcal{D}(\mathcal{G})\subseteqq\mathcal{D}(\mathcal{G}^{(n)})$ *and that* (2) *for any fixed x and for any $\varepsilon>0$, there exists some compact set K such that*

$$G_0^{(n)}(x, K^c)<\varepsilon \quad \text{for every } n. \tag{3.1}$$

Moreover suppose that (a) $u\in\mathcal{D}(\mathcal{G})$, (b) $\mathcal{G}^{(n)}u$ *is uniformly bounded and* (c) $\mathcal{G}^{(n)}u$ *converges to $\mathcal{G}u$, uniformly on any compact set. Then, for $f=G_\alpha^{-1}u$, we have*

$$\lim_{n\to\infty} G_\alpha^{(n)}f=u(=G_\alpha f). \tag{3.2}$$

PROOF. Since $u\in\mathcal{D}(\mathcal{G}^{(n)})$, there exists $f^{(n)}=[G_\alpha^{(n)}]^{-1}u$ and we have $\mathcal{G}^{(n)}u=\alpha u-f^{(n)}$. By (b) and (c), $f^{(n)}$ is uniformly bounded in n and $f^{(n)}$ converges to f, uniformly on any compact set. Then it follows that, for a compact set K,

$$\begin{aligned}|G_\alpha f(x)-G_\alpha^{(n)}f(x)|&=|G_\alpha^{(n)}[f^{(n)}-f](x)|\\&\leqq \sup_{y\in K}|f^{(n)}(y)-f(y)|G_\alpha^{(n)}(x,K)+AG_\alpha^{(n)}(x,K^c),\\&\leqq\frac{1}{\alpha}\sup_{y\in K}|f^{(n)}(y)-f(y)|+AG_0^{(n)}(x,K^c),\end{aligned}$$

where $A=\sup_{x,n}|f^{(n)}(x)|+\sup_x|f(x)|$. Taking a sufficiently large K and letting $n\to\infty$, the last expression can be made arbitrarily small.

Now let $\mathcal{U}$ be an open base and ρ, any metric of S. For each n, we can choose the systems $\{U_i^{(n)}; i=1,2,\cdots\}$, $\{V_i^{(n)}; i=1,2,\cdots\}$ of sets in $\mathcal{U}$ satisfying the following conditions. (1) Each $\overline{U_i^{(n)}}$ is compact and $d(U_i^{(n)})<\frac{1}{n}$.[8] (2) $\overline{V_i^{(n)}}\subset U_i^{(n)}$ for every i. (3) $\bigcup_i V_i^{(n)}=S$. (4) For any compact set K, only the finite number of $V_i^{(n)}$'s intersect with K. We now define $\sigma_k^{(n)}$ by

$$\begin{aligned}\sigma_1^{(n)}(w)&=\sigma_{\overline{U_i^{(n)}}^c}(w) \quad \text{if} \quad x_0(w)\in V_i^{(n)}-V_i^{(n)}\cap\Big[\bigcup_{j=1}^{i-1}V_j^{(n)}\Big],\\ \sigma_k^{(n)}(w)&=\sigma_{k-1}^{(n)}(w)+\sigma_1^{(n)}(w^+_{\sigma_{k-1}^{(n)}}) \quad \text{for } k\geqq 2.\end{aligned} \tag{3.3}$$

LEMMA 3.2. (i) $\sigma_1^{(n)}$ (*and therefore every* $\sigma_k^{(n)}$) *is a Markov time.* (ii) $\lim_{k\to\infty}\sigma_k^{(n)}\geqq\sigma_\infty$.

PROOF. Noting that every open set is admissible, (i) is clear from the formula

$$\{\sigma_1^{(n)}(w)\geqq t\}=\bigcup_i\Big\{\sigma_{\overline{U_i^{(n)}}^c}(w)\geqq t,\ x_0(w)\in V_i^{(n)}-V_i^{(n)}\cap\Big[\bigcup_{j=1}^{i-1}V_j^{(n)}\Big]\Big\}.$$

To prove (ii), assume that $\lim_{k\to\infty}\sigma_k^{(n)}(w)<\sigma_\infty(w)$ for some w. Then, from the definition of W (see (I)), $\lim_{k\to\infty}x_{\sigma_k^{(n)}(w)}(w)$ exists in S and the trajectory $\{x_t(w); t\leqq\lim_{k\to\infty}\sigma_k^{(n)}(w)\}$ is contained in some compact set $K(w)$.

8) $d(U_i^{(n)})$ is the ρ-diameter of $U_i^{(n)}$, that is, $\sup_{x,y\in U_i^{(n)}}\rho(x,y)$.

Putting $\delta=\inf_{V_i^{(n)}\cap K(w)\neq\phi}\rho(V_i^{(n)},\overline{U_i^{(n)c}})>0$, we have

$$\delta<\rho(x_{\sigma_k^{(n)}(w)}(w),\ x_{\sigma_{k+1}^{(n)}(w)}(w))\to 0\quad(k\to\infty),$$

which is a contradiction.

LEMMA 3.3. *Let X be a strict Markov process such that G_α $(\alpha>0)$ $\boldsymbol{C}(S)\to\boldsymbol{C}(S)$ and*

(3.4) $$\sup_{x\in S}G_0(x,S)<+\infty.$$

(i) *For each n, $q^{(n)}(x)=[E_x(\sigma_1^{(n)})]^{-1}$ and $\prod^{(n)}(x,A)=P_x(x_{\sigma_1^{(n)}}\in A)$ satisfy the conditions in Proposition* 3.2 *of* (I). *The regular step process corresponding to $(q^{(n)},\prod^{(n)})$ is denoted by $X^{(n)}$ and the Green operator of $X^{(n)}$, by $G_\alpha^{(n)}$.* (ii) *$X^{(n)}$ approximates X in the following sense:*

(3.5) $$\lim_{n\to\infty}G_\alpha^{(n)}f=G_\alpha f\quad \textit{for every } \alpha>0 \textit{ and every } f\in\boldsymbol{C}(S).$$

REMARK. It is easily verified that (3.5) implies

(3.6) $$\liminf_{n\to\infty}G_\alpha^{(n)}f\geqq G_\alpha f\quad(\alpha>0)$$

for every lower semicontinuous function $f\geqq 0$.

PROOF. (i) A simple calculation shows that,

$$[q^{(n)}(x)]^{-1}=E_x(\sigma_{\overline{U_i^{(n)}}^c})=E_x\left(\int_0^{\sigma_{\overline{U_i^{(n)}}^c}}\chi_{\overline{U_i^{(n)}}}(x_t)\,dt\right)^{9)}$$
$$\leqq G_0(x,S)<+\infty\quad\text{for}\quad x\in V_i^{(n)}-V_i^{(n)}\cap\left[\bigcup_{j=1}^{i-1}V_j^{(n)}\right],$$

which proves $q^{(n)}(x)>0$. It is self-evident that the other conditions for the canonical system are satisfied by $(q^{(n)},\prod^{(n)})$.

(ii). From Lemma 3.2. (ii),

$$G_0(x,S)=E_x\left(\int_0^{\sigma_\infty}\chi_S(x_t)\,dt\right)=\sum_{k\geqq 0}E_x\left(\int_{\sigma_k^{(n)}}^{\sigma_{k+1}^{(n)}}\chi_S(x_t)\,dt\right).$$
$$=\sum_{k\geqq 0}E_x\{E_{x_{\sigma_k^{(n)}}}(\sigma_1^{(n)})\}=\sum_{k\geqq 0}E_x([q^{(n)}(x_{\sigma_k^{(n)}})]^{-1})$$
$$=\sum_{k\geqq 0}[\prod^{(n)}]^k[q^{(n)}]^{-1}(x)=G_0^{(n)}(x,S).$$

According to (3.4), $G_0^{(n)}$ is a bounded operator as well as G_0. Therefore, $\mathcal{D}(\mathcal{G})=G_0(\boldsymbol{D}_X(S))$ and $\mathcal{D}(\mathcal{G}^{(n)})=G_0^{(n)}(\boldsymbol{D}_{X^{(n)}}(S))$. Moreover $\boldsymbol{D}_{X^{(n)}}(S)=\boldsymbol{B}(S)$, because $X^{(n)}$ is a regular step process. Consequently, in order to prove $\mathcal{D}(\mathcal{G})\subseteqq\mathcal{D}(\mathcal{G}^{(n)})$, it is enough to show that, for any $f\geqq 0$ of $\boldsymbol{B}(S)$, G_0f is a $G_0^{(n)}$-potential. But

$$+\infty>G_0f(x)\geqq[\prod^{(n)}]^kG_0f(x)=E_x\left(\int_{\sigma_k^{(n)}}^{\sigma_\infty}f(x_t)\,dt\right)\to 0,$$

so that G_0f is a $G_0^{(n)}$-potential, according to Proposition 3.6 of (I).[10] Next to show (3.1), take a compact set K such that $G_0(x,\widetilde{K}^c)<\varepsilon$ and put $A^{(n)}=\bigcup_{U_i^{(n)}\cap\widetilde{K}\neq\phi}U_i^{(n)}$. It is immediate from the definition of $U_i^{(n)}$ that

9) $\chi_A(\cdot)$ denotes the indicator function of the set A.

$A^{(n)}$ is contained in some compact set K. By simple consideration, we get

$$E_x\left(\int_{\sigma_k^{(n)}}^{\sigma_{k+1}^{(n)}} \chi_{\tilde{K}^c}(x_t)\,dt\right) \geqq E_x\left\{\chi_{K^c}(x_{\sigma_k^{(n)}})\int_{\sigma_k^{(n)}}^{\sigma_{k+1}^{(n)}} \chi_S(x_t)\,dt\right\}$$
$$= E_x\{\chi_{K^c}(x_{\sigma_k^{(n)}})[q^{(n)}(x_{\sigma_k^{(n)}})]^{-1} = [\Pi^{(n)}]^k[q^{(n)}]^{-1}\chi_{K^c}(x).$$

Therefore

$$\varepsilon > G(x, \tilde{K}^c) = \sum_{k\geqq 0} E_x\left(\int_{\sigma_k^{(n)}}^{\sigma_{k+1}^{(n)}} \chi_{\tilde{K}^c}(x_t)\,dt\right)$$
$$\geqq \sum_{k\geqq 0} [\Pi^{(n)}]^k[q^{(n)}]^{-1}\chi_{K^c}(x) = G_0^{(n)}(x, K^c).$$

Thus we see that our processes X, $X^{(n)}$ satisfy the conditions (1), (2) of Lemma 3.1.

Now take any $f \in \boldsymbol{C}(S)$ and put $u = G_\alpha f$. Since $u \in \boldsymbol{C}(S)$ from the assumption, $\mathcal{G}u = \alpha u - f$ is also bounded and continuous. Noting that $u \in \mathcal{D}(\mathcal{G}^{(n)})$ and recalling the remark of Proposition 2.1, we have

$$\mathcal{G}^{(n)}u(x) = \mathcal{G}_D^{(n)}u(x) = \frac{E_x(u(x_{\sigma_1^{(n)}})) - u(x)}{E_x(\sigma_1^{(n)})}.$$

Consequently it follows from (2.1) that, if $x \in V_i^{(n)} - V_i^{(n)} \cap \left[\bigcup_{j=1}^{i-1} V_j^{(n)}\right]$,

$$|\mathcal{G}u(x) - \mathcal{G}^{(n)}u(x)| = \left|\mathcal{G}u(x) - \frac{E_x(u(x_{\sigma_{\overline{U_i^{(n)}}^c}})) - u(x)}{E_x(\sigma_{\overline{U_i^{(n)}}^c})}\right|$$
$$\leqq \sup_{y \in \overline{U_i^{(n)}}} |\mathcal{G}u(x) - \mathcal{G}u(y)|,$$

which implies that u satisfies the conditions (b), (c) in Lemma 3.1. Therefore we can apply Lemma 3.1 for any $f \in \boldsymbol{C}(S)$ and our lemma is proved completely.

4. Proof of Theorem 1. Let X be the process of Theorem 1 and $X^{(\beta)}$ $(\beta > 0)$, the β-subprocess of X. Moreover suppose that u is lower semicontinuous and superharmonic with respect to the open base $\mathcal{U}$($\mathcal{U}$-superharmonic). Clearly u is $\mathcal{U}$-superharmonic for $X^{(\beta)}$.[11] The n-th regular step process of Lemma 3.3 induced by the base $\mathcal{U}$ is denoted by $X^{(\beta,n)}$. From the definition of $X^{(\beta,n)}$, u is $\Pi^{(\beta,n)}$-superharmonic. Therefore, $u \geqq \alpha G_\alpha^{(\beta,n)}u$, according to Theorem of (I). On the other hand, since $X^{(\beta)}$ satisfies the assumptions of Lemma 3.3, $X^{(\beta,n)}$ approximates $X^{(\beta)}$ in the sense of (3.5). Noting that u is lower semicontinuous, it follows from the remark of Lemma 3.3 that $u \geqq \liminf_{n\to\infty} \alpha G_\alpha^{(\beta,n)}u \geqq \alpha G_\alpha^{(\beta)}u = \alpha G_{\alpha+\beta}u$. Letting $\beta \to 0$, we have $u(x) \geqq \alpha G_\alpha u(x)$.

---

10) Added in proof: We have just proved that every $u \in \mathcal{D}(\mathcal{G})$ can be written in the form $u = G_0^{(n)}f^{(n)}$. We can easily show that $f^{(n)} \in \boldsymbol{B}(S)$, using (2.1).

11) In general, it is known that $E_x^{(\beta)}(u(x_\sigma)) = E_x(e^{-\beta\sigma}u(x_\sigma))$ for any Markov time σ.

Considering the lower semicontinuity of u, we get

$$u(x) \geqq \liminf_{\alpha\to\infty} \alpha G_\alpha u(x) \geqq E_x\left(\int_0^\infty e^{-t} \liminf_{\alpha\to\infty} u(x_{t/\alpha})dt\right) \geqq u(x).$$

Consequently, according to Proposition 2.4 of (I), u is excessive.

5. Some further results. It should be noted that Theorem 1 does not contain directly the fact that any nonnegative superharmonic function in the usual sense is an excessive function associated with the Brownian motion. To cover such case or others, we shall take another family as $\mathcal{F}$ and derive a similar result to Theorem 1.

Let ρ be a metric of S and $U(x, r)$, the ball $\{y; \rho(x, y) < r\}$. Next define $\mathcal{F}_x^\rho = \{\overline{U(x, r)}^c; r > 0$ and $\overline{U(x, r)}$ is compact$\}$ and $\mathcal{F}^\rho = \bigcup_{x\in S} \mathcal{F}_x^\rho$. Moreover take a sequence of positive $\mathcal{B}$-measurable functions $r^{(n)}(x)$ such that (1) $U(x, r^{(n)}(x)) \in \mathcal{F}_x^\rho$, (2) $\inf_{x\in K} r^{(n)}(x) > 0$ (for any compact K) and (3) $\sup_{x\in S} r^{(n)}(x) \downarrow 0$ $(n\to\infty)$. Then it is shown that $\tau_1^{(n)}(w) = \sigma_{\overline{U(x_0(w), r^{(n)}_{(x_0(w))})}^c}(w)$ (and therefore every $\tau_k^{(n)}(w) = \tau_{k-1}^{(n)}(w) + \tau_1^{(n)}(w^+_{\tau_{k-1}^{(n)}}))$ is a Markov time and that such $\tau_k^{(n)}$ acts as a substitute for $\sigma_k^{(n)}$ in Lemma 3.3. Therefore the argument of Section 4 is applicable for $\mathcal{F}^\rho$-superharmonic functions and we get

THEOREM 2. *Let X be the process of Theorem* 1. *Then, if u is lower semicontinuous and $\mathcal{F}^\rho$-superharmonic for some metric ρ, it is excessive.*

In particular, if G_α maps the family $\boldsymbol{B}(S)$ of all bounded $\mathcal{B}$-measurable functions into $\boldsymbol{C}(S)$, any excessive function is lower semicontinuous. Combining Proposition 2.4 of (I), Theorem 1 and Theorem 2, we have

THEOREM 3. *Suppose that X is a strict Markov process such that G_α $(\alpha>0)$: $\boldsymbol{B}(S)\to\boldsymbol{C}(S)$. Then a function u is excessive if and only if it is lower semicontinuous and superharmonic (or $\mathcal{F}^\rho$-superharmonic for some ρ.*

References

[1] E. B. Dynkin: Intrinsic topology and excessive functions associated with Markov process, Dokl. Akad. Nauk SSSR (in Russian), **127**, 17-19 (1959).

[2] E. B. Dynkin: Theory of Markov Processes, London (1960).

[3] K. Itô: Lectures on Markov Processes, Tata Institute, to appear.

[4] T. Watanabe: On the equivalence of excessive functions and superharmonic functions in the theory of Markov Processes. I, Proc. Japan Acad., **38**, (1962).