222. *A Note on Function-theoretic Null-sets of Class $N_{\mathfrak{S}\mathfrak{D}}$*

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1. Let Ω be a plane region and let E be its complementary set with respect to the Riemann sphere. Following Ahlfors and Beurling [1], we shall say that E is of class $N_{\mathfrak{B}}$ (resp. $N_{\mathfrak{D}}$, $N_{\mathfrak{S}\mathfrak{B}}$, and $N_{\mathfrak{S}\mathfrak{D}}$), if Ω carries no non-constant analytic function in it which is bounded (resp. with finite Dirichlet integral, univalent and bounded, and univalent and with finite Dirichlet integral). They showed that $N_{\mathfrak{B}} \subsetneqq N_{\mathfrak{D}} \subsetneqq N_{\mathfrak{S}\mathfrak{B}} = N_{\mathfrak{S}\mathfrak{D}}$.

It is known that the union of a finite number of sets of class $N_{\mathfrak{B}}$ (resp. $N_{\mathfrak{D}}$) belongs to the same class. This was proved by Kametani [4] for mutually disjoint sets of class $N_{\mathfrak{B}}$ and later by Kuroda [5] for the same class without the restriction of disjointness. This is also true for the union of a countable number of these sets, so long as it is compact (see Noshiro [6], footnote, p. 11).

The class $N_{\mathfrak{S}\mathfrak{D}}$ does not have this property. We shall verify this by constructing a counterexample.

2. A boundary component α of a plane region is called *weak* [8], if its image under every conformal mapping is always a point. Among several properties of weak boundary components obtained by Grötzsch [2], Sario [7, 8], Jurchescu [3], and others, the following will be needed in the next section;

i) (Jurchescu [3]) The weakness is a boundary property. That is, if there exists a conformal mapping, denoted by $f(z)$, of a neighborhood of a weak boundary component α onto a neighborhood of $f(\alpha)$ of a region, then $f(\alpha)$ is weak with respect to the region.

ii) (Sario [7] and Jurchescu [3]) A compact set E is of class $N_{\mathfrak{S}\mathfrak{D}}$, if and only if each boundary component of E^c is weak.

iii) (Grötzsch [2]) α is weak, if there exists a sequence of mutually disjoint doubly connected regions $\{R_n\}$ such that R_{n+1} separates α from R_n and that the series $\sum \bmod R_n$ diverges. Here $\bmod R_n$ is the logarithm of the ratio of the outer and the inner radius of a conformally equivalent annulus of R_n.

By the properties i) and ii) we get immediately

Lemma. *The union of a finite number of mutually disjoint sets of class $N_{\mathfrak{S}\mathfrak{D}}$ is in the same class.*

3. We now construct our example, stating

Theorem. *There exist two compact sets of class $N_{\mathfrak{S}\mathfrak{D}}$, whose union is not of class $N_{\mathfrak{S}\mathfrak{D}}$.*

Proof. *Let $\{A_n\}$ be a sequence of annuli,*

$$A_n:\ 2^n<|z|<2^{n+1}(n=0, 1, \cdots).$$

There exists a compact set, denoted by e, of class $N_{\mathfrak{S}\mathfrak{D}}$ and with positive area, say k [1]. We may assume that it is contained in the open square S: $|\mathrm{Re}\, z|<1/2$ and $|\mathrm{Im}\, z|<1/2$. We shall insert such a set e_n of class $N_{\mathfrak{S}\mathfrak{D}}$ into each A_n that the area of the region A_n-e_n is less than $1/2^n$. To this end, using a net system with sides parallel to the axes, we can express the region A_n as the union of a sequence of closed non-overlapping squares, denoted by $\{W_n^{(\nu)}\}$. Each square $W_n^{(\nu)}$ has a linear transformation from S onto it, denoted by $1_n^{(\nu)}$. Put $1_n^{(\nu)}(e)=e_n^{(\nu)}$, which is clearly of class $N_{\mathfrak{S}\mathfrak{D}}$. Take a sufficiently large number of $e_n^{(\nu)}$'s so that the area

$$m\left(A_n-\bigcup_{\nu=0}^{N_1} e_n^{(\nu)}\right)<h\, m(A_n),$$

where h is a fixed positive number less than $1-k$ and $m(*)$ denotes the Lebesgue area.

Applying this procedure to the region $A_n-\bigcup_{\nu=0}^{N_1} e_n^{(\nu)}$ and so on, we can take a finite number of mutually disjoint sets of class $N_{\mathfrak{S}\mathfrak{D}}$, whose union, denoted by e_n, is of class $N_{\mathfrak{S}\mathfrak{D}}$ by the lemma and satisfying that $m(A_n-e_n)<1/2^n$.

We set $E_1=\overline{\bigcup_{n=0}^{\infty} e_{2n}}$ and $E_2=\overline{\bigcup_{n=0}^{\infty} e_{2n+1}}$ which are both of class $N_{\mathfrak{S}\mathfrak{D}}$ and whose intersection is clearly the point at infinity. In fact, for example, every component of E_1 other than the point at infinity is weak by the lemma and the property i) while so is the point at infinity by the property iii), since $\sum_{n=0}^{\infty} \mathrm{mod}\, A_{2n+1}=\infty$, where $\mathrm{mod}\, A_{2n+1}=\log 2\ (n=0, 1, \cdots)$.

On the other hand the union $E_1\cup E_2$ is not of class $N_{\mathfrak{S}\mathfrak{D}}$ by the definition. Indeed the area of its complementary set, which is a region, is finite and the function z is univalent and with finite Dirichlet integral in the region. Thus we complete the proof.

4. This example shows that the union of mutually disjoint but a countable number of sets of class $N_{\mathfrak{S}\mathfrak{D}}$ is not, in general, so, even if it is compact. Indeed, the sets e_n and the point at infinity are the desired.

References

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