74. *Quasi-Conformal Extension of Meier's Theorem*

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We shall show that the so-called Meier's topological analogue of Plessner's theorem ([5], Satz 5, cf. [2], p. 154) is true of quasi-conformal functions in $U:|z|<1$. A function $w=f(z)$ defined in a plane domain G with its values in the w-sphere $\Omega:|w|\leqq\infty$ is called quasi-conformal (precisely, K-quasi-conformal) in G if f is of the composed form: $g\circ T(z)$, where $\zeta=T(z)$ is a K-quasi-conformal homeomorphism from G onto another plane domain G' and $w=g(\zeta)$ is meromorphic in G' (cf. [4], p. 250).

Let $f(z)$ be a quasi-conformal function in U and let $e^{i\theta}$ be a point of $\Gamma:|z|=1$. Then the cluster set $C(f,e^{i\theta})$, an angular cluster set $C_\Delta(f,e^{i\theta})$ and a chordal cluster set $C_{\rho(\varphi)}(f,e^{i\theta})$ are defined in the same manner as in [2] (pp. 1, 73 and 72), where Δ is the interior of a triangle in U with one vertex $e^{i\theta}$ (simply, "angle Δ at $e^{i\theta}$") and $\rho(\varphi)$ is a chord of Γ passing through $e^{i\theta}$ and making a directed angle φ, $|\varphi|<\pi/2$, with the radius to $e^{i\theta}$. A point $e^{i\theta}\in\Gamma$ is a Plessner point of f if $C_\Delta(f,e^{i\theta})=\Omega$ for any angle Δ at $e^{i\theta}$. A point $e^{i\theta}\in\Gamma$ is a Meier point of f if $C(f,e^{i\theta})\neq\Omega$ and $C_{\rho(\varphi)}(f,e^{i\theta})=C(f,e^{i\theta})$ for all φ, $|\varphi|<\pi/2$. We denote by $I(f)$ ($M(f)$, resp.) the set of all Plessner points (Meier points, resp.) of f.

We first prove a topological analogue of Fatou's theorem (cf. [5], Satz 6, [2], p. 154).

Theorem 1. *Let f be a bounded quasi-conformal function in U. Then $\Gamma\backslash M(f)$ is of first Baire category on Γ.*

Proof. We shall use the Schwarz lemma for quasi-conformal functions (cf. [3]) in the following form: Let $h(z)$ be a K-quasi-conformal function in the disk $\delta(z_o,q):|z-z_o|<q$. If $|h(z)|<M, M>0$ being a constant, in $\delta(z_o,q)$, then

(1) $$|h(z)-h(z_o)|\leqq 8Mq^{-1/K}|z-z_o|^{1/K},\qquad z\in\delta(z_o,q).$$

We let, for the proof, $h(z)=g\circ T(z)$, where T is a K-quasi-conformal self-homeomorphism of $\delta(z_o,q)$ with $z_o=T(z_o)$, which we may suppose, and g is holomorphic in $\delta(z_o,q)$. Then, Theorem 5, (9) of Mori [6] reads

$$|T(z)-T(z_o)|\leqq 4q^{1-(1/K)}|z-z_o|^{1/K}.$$

Combined with the Schwarz lemma for the bounded g, this gives (1).

Let $e^{i\theta} \in \Gamma \backslash M(f)$. Then we have a chord $\rho(\varphi)$ at $e^{i\theta}$ with $C_{\rho(\varphi)}(f, e^{i\theta}) \neq C(f, e^{i\theta})$, where we may suppose $0 \leqq \varphi < \pi/2$. Let $P \in C(f, e^{i\theta}) \backslash C_{\rho(\varphi)}(f, e^{i\theta})$ and let $\bar{\delta}(P, 2\beta)$ be a closed disk with the centre $P \neq \infty$ and the radius $2\beta < (1/2) \operatorname{dis}\{C_{\rho(\varphi)}(f, e^{i\theta}), P\}$. Then we may find a segment $\rho_1(\varphi) \subset \rho(\varphi)$, one end-point of which is $e^{i\theta}$, such that

$$(2) \qquad \overline{f(\rho_1(\varphi))} \cap \bar{\delta}(P, 2\beta) = \emptyset \text{ (empty)}$$

and that $\gamma(\xi) < 1 - |\xi|$ for $\xi \in \rho_1(\varphi)$, where $\gamma(\xi) = |\xi - e^{i\theta}| \sin(\pi/4 - \varphi/2)$, the latter being a consequence of: $\gamma(\xi) \to 0$ as $\rho(\varphi) \ni \xi \to e^{i\theta}$. By boundedness of f, we have $|f(z)| < m/8$, $m > 0$ being a constant, $z \in U$. We set $\gamma_o = \min\{(\beta/m)^K, 1\}$ and we let A_ξ, $\xi \in \rho_1(\varphi)$, be the open disk: $|z - \xi| < \gamma_o \gamma(\xi)$. It follows from (1) with $h(z) = f(z)$, $z_o = \xi$, $q = \gamma(\xi)$, $M = m/8$, that

$$(3) \qquad |f(z) - f(\xi)| \leqq \beta$$

for any $\xi \in \rho_1(\varphi)$ and $z \in A_\xi$ since A_ξ is contained in $|z - \xi| < \gamma(\xi)$. It follows from (2) and (3) that

$$(4) \qquad \overline{f(A_\xi)} \cap \bar{\delta}(P, \beta/2) = \emptyset$$

for $\xi \in \rho_1(\varphi)$. Now, as $\rho_1(\varphi) \ni \xi \to e^{i\theta}$, the disks A_ξ sweep an angle Δ at $e^{i\theta}$ bisected by $\rho(\varphi)$, so that by (4) we have

$$\overline{f(\Delta)} \cap \bar{\delta}(P, \beta/2) = \emptyset$$

and hence

$$C_\Delta(f, e^{i\theta}) \neq C(f, e^{i\theta}).$$

Our theorem follows from Collingwood's maximality theorem ([1], Theorem 4, cf. [2], p. 80). Q.E.D.

Let γ be an arbitrary simple arc in U terminating at $e^{i\theta}$ and tangent at $e^{i\theta}$ to a chord $\rho(\varphi)$ at $e^{i\theta}$. Then the curvilinear cluster set $C_\gamma(f, e^{i\theta})$ ([2], p. 72) coincides with $C_{\rho(\varphi)}(f, e^{i\theta})$ if f is bounded and quasi-conformal in U. For the proof, we use the same method as in the proof of Theorem 1.

Following the familiar lines ([5], cf. [2], p. 155) we have

Theorem 2. *Let $f(z)$ be a quasi-conformal function in U. Then $\Gamma \backslash \{M(f) \cup I(f)\}$ is of first Baire category on Γ.*

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