176. *On the Williamson's Conjecture*

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1. Let G be a non-discrete locally compact abelian group with the dual group Γ of G. We will denote by $M(G)$ the Banach algebra of all bounded regular Borel measures on G under the convolution multiplication.

It is known that there exists a compact commutative topological semigroup S and an order preserving isometric-isomorphism θ of $M(G)$ into $M(S)$ such that:

(a) if $\mu \in M(G)$, $\nu \in M(S)$ and ν is absolutely continuous with respect to $\theta\mu$, then there is $\omega \in M(G)$ such that $\theta\omega = \nu$: and

(b) each multiplicative linear functional h on $M(G)$ has the form

$$h(\mu) = \int_S f d\theta\mu \qquad (\mu \in M(G))$$

for an unique nonzero continuous semicharacter f on S (cf. [4]).

The set of all nonzero continuous semicharacters on S is denoted by $\hat{S}$. We may consider $\hat{S}$ to be the maximal ideal space of $M(G)$. Furthermore, $\hat{S}$ is a compact semigroup and Γ may be considered to be the maximal group at the identity of $\hat{S}$ (cf. [5]).

We denote by Δ the subset of $\hat{S}$ consisting of functionals symmetric in the sense that $\hat{\mu}^*(f) = \overline{\hat{\mu}(f)}$ for any $\mu \in M(G)$, where * denotes the usual involution on $M(G)$. Let $M(\Delta) = \{\mu \in M(G) : \hat{\mu}(f) = 0$ for all $f \in \hat{S} \backslash \Delta\}$.

Let $M_c(G)$ denote the algebra of all continuous measures of $M(G)$.

Our purpose is to show that the following *Williamson's conjecture* (cf. [6]) is true.

Williamson's conjecture: *If $\mu \in M(\Delta)$, then $\mu \in M_c(G)$.*

2. By (a), if $f \in \hat{S}$ and $\mu \in M(G)$, then there is a measure $\mu_f \in M(G)$ such that $d\theta\mu_f = f d\theta\mu$.

The following lemmas are essential to prove that Williamson's conjecture is true.

Lemma 1. *If $f \in \Gamma$ and $\mu \in M(G)$, then $d\theta\mu_f^* = f d\theta\mu^*$.*

Lemma 2. *If $f \in \Gamma$ and $g \in \hat{S} \backslash \Delta$, then $fg \in \hat{S} \backslash \Delta$.*

For any $f \in \hat{S}$, let $S_f = \{s \in S : f(s) \neq 0\}$ and let $J_f = \{s \in S : f(s) = 0\}$.

Theorem 3. *If $g \in \hat{S} \backslash \Delta$ and $\mu \in M(\Delta)$, then $\theta\mu|_{S_g}$, the restriction to S_g of $\theta\mu$, is zero measure. In particular, $M(\Delta) \subset M_c(G)$.*

From this, it follows that:

Corollary 4. *Let* $J_\Delta=\{s\in S: f(s)=0$ *for all* $f\in\Delta\}$, *then* $M(\Delta)=\{\mu\in M(G): \theta\mu$ *is concentrated on* $J_\Delta\}$. *Thus,* $M(\Delta)$ *is an L-ideal.*

If A is a subset of $M(G)$ such that $\mu\in A$ implies $\mu^*\in A$, then we call A symmetric.

Lemma 5*). *Let* f *be an idempotent semicharacter on* S *and let* $I=\{\mu\in M(G): \theta\mu$ *is concentrated on* $J_f\}$, *then the following statements are equivalent:*

(1) $f\in\Delta$;

(2) $f\cdot\Gamma\subset\Delta$;

(3) *I is a symmetric prime L-ideal* (cf. [3]).

From this lemma, we have the next theorem.

Theorem 6. *If I is a non-symmetric prime L-ideal, then* $M(\Delta)\subset I$.

This theorem have the following result.

Corollary 7. $M(\Delta)$ *is a symmetric L-ideal.*

At last, we will state two special results.

Let $\mathfrak{T}$ be a *Raikov system* (cf. [7]), then we denote by $M(\mathfrak{T})$ the closed subalgebra of $M(G)$ consisting of measures that are concentrated on Raikov system $\mathfrak{T}$. We denote by $M(\mathfrak{T})^\perp$ the complementary ideal of $M(\mathfrak{T})$, consisting of all measures that are singular with respect to all measures in $M(\mathfrak{T})$.

Theorem 8. *Let G be a metric I-group. Let P is a compact independent perfect subset of G and let* $\mathfrak{T}$ *be the Raikov system generated by* $P\cup(-P)$. *Then,* $M(\Delta)\subset M(\mathfrak{T})^\perp$.

Theorem 9. *If H is a non-discrete, non-open and closed subgroup of G. Let* $\mathfrak{A}=\{\bigcup_{i=1}^{\infty}(x_i+H): x_i\in G,\ i=1,2,3,4,\cdots\}$. *Let* $M(\mathfrak{A})$ *be the closed subalgebra of* $M(G)$ *consisting of all measures that are concentrated on* $\mathfrak{A}$. *Then,* $M(\Delta)\subset M(\mathfrak{A})^\perp$.

The detail of the proof will appear in the nearest future.

References

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*) This lemma shows that there is a gap between the proof in [1] and ours.