226. On Realization of the Discrete Series for Semisimple Lie Groups

By Ryoshi HOTTA

(Comm. by Kunihiko KODAIRA, M. J. A., Nov. 12, 1970)

This note is an announcement of a result, which says, briefly, that most of the discrete series for a semisimple Lie group are realized as certain eigenspaces of the Casimir operator on the symmetric space (Theorem 2). This construction is in some sense a generalization of the methods adopted in [1], [2], [9] for special groups and in [5] for the groups of hermitian type. Also, [6] indicates the above method of realization. Further, as for alternative methods to realize most of the discrete series, we refer to the recent works [5], [8]. Our technique used here depends heavily on that of [5]. A detailed exposition with full proofs will appear elsewhere.

1. Let G be a connected non-compact semisimple Lie group with a compact Cartan subgroup. We assume, for convenience, that G has a faithful finite dimensional representation and its complexification G^C is simply connected. Fix a maximal compact subgroup K of G and a Cartan subgroup H contained in K. We denote by $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{h}$ the Lie algebras corresponding to G, K and H respectively. For complexifications $\mathfrak{g}^C, \mathfrak{k}^C, \mathfrak{h}^C$ of $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}$, we denote by Δ the root system of $(\mathfrak{g}^C, \mathfrak{h}^C)$, and by W_G the Weyl group of $(\mathfrak{k}^C, \mathfrak{h}^C)$. Taking a positive root system P of Δ fixed once for all, P_k (resp. P_n) denotes the set of a positive compact (resp. non-compact) roots. Let L be the character group of H, L' the set of regular elements in L. Introducing an inner product (,) on L induced by the Killing form, we put $\varepsilon(\lambda) = \text{sign} \prod_{\alpha \in P} (\lambda, \alpha)$ for $\lambda \in L'$, and $\varepsilon(\lambda)=0$ for $\lambda \in L-L'$. We also put $\varepsilon_k(\lambda) = \text{sign} \prod_{\alpha \in P_k} (\lambda, \alpha)$ if $\lambda \in L$ is $\mathfrak{k}^C$-regular, and $\varepsilon_k(\lambda)=0$ if λ is $\mathfrak{k}^C$-singular. For discrete series, the following fact is known by Harish-Chandra [3]. Let $\mathcal{E}_d$ be the discrete series for G. For $\lambda \in L'$, there then exists a unique element $\omega(\lambda) \in \mathcal{E}_d$, and the map $L' \ni \lambda \mapsto \omega(\lambda) \in \mathcal{E}_d$ is surjective, while $\omega(\lambda)=\omega(\lambda')$ if and only if there exists $w \in W_G$ such that $w\lambda = \lambda'$. We shall denote by $\Theta_{\omega(\lambda)}$ the character of $\omega(\lambda)$.

For a finite subset A of L, we shall denote by $|A|$ its cardinal number and put $\langle A \rangle = \sum_{\alpha \in A} \alpha$. Put $\rho = \langle P \rangle /2, \rho_k = \langle P_k \rangle /2$ and $\rho_n = \rho - \rho_k$. If $\varepsilon_k(\lambda + \rho_k) \neq 0$ for $\lambda \in L$, there exists a unique $w \in W_G$ such that $w(\lambda+\rho_k)-\rho_k$ is k^C-dominant. We then denote by $[\lambda]$ the equivalence class to which belongs an irreducible K-module with highest weight

$w(\lambda+\rho_k)-\rho_k$. For the sake of notational convenience, we put $[\lambda]=0$ if $\varepsilon_k(\lambda+\rho_k)=0$. We shall denote by $\chi(\lambda)$ the character of $[\lambda]$.

2. For a finite dimensional unitary K-module V, we denote by $\mathcal{V}$ the homogeneous vector bundle over G/K associated to V, whose fiber has an invariant hermitian metric. Throughout this note, for a K-module the corresponding script letter denotes the homogeneous vector bundle associated to the K-module given. Let $L_2(\mathcal{V})$ (resp. $C^\infty(\mathcal{V})$) be a space of all square-integrable (resp. differentiable) sections of $\mathcal{V}$, which is naturally regarded as the space consisting of all V-valued square-integrable (resp. differentiable) functions f satisfying $f(gk)=k^{-1}f(g)$ for $k\in K, g\in G$. Now assume that there are given two K-modules V, W. For a G-invariant linear differential operator $D: C^\infty(\mathcal{V})\to C^\infty(\mathcal{W})$, the maximal extension $D: L_2(\mathcal{V})\to L_2(\mathcal{W})$ means the closed linear operator whose domain consists of $f\in L_2(\mathcal{V})$ such that $Df\in L_2(\mathcal{W})$ in the sense of distributions. We shall hereafter consider differential operators on square-integrable sections in this sense. Let $D^*: L_2(\mathcal{W})\to L_2(\mathcal{V})$ be the maximal extension of the formal adjoint operator for D. We then have the unitary representations of G on the Hilbert spaces $\operatorname{Ker} D$ and $\operatorname{Ker} D^*$. Let $(\operatorname{Ker} D)_d$ (resp. $(\operatorname{Ker} D^*)_d$) be the smallest closed invariant subspace which contains every irreducible closed invariant subspace of $\operatorname{Ker} D$ (resp. $\operatorname{Ker} D^*$). Denote by π_V (resp. π_W) the representation on the space $(\operatorname{Ker} D)_d$ (resp. $(\operatorname{Ker.} D^*)_d$). It is then shown that the operator $\pi_V(\varphi)=\int_G \varphi(g)\pi_V(g)dg$ is of trace class for a compactly supported C^∞-function φ on G, and so defines an invariant distribution $\operatorname{Trace}\pi_V$ on G (the same holds also for π_W). The following theorem can be proved by a similar method to the one in [5].

Theorem 1. *Under the above situation, assume that D is at most a first order operator, and denote by χ_V, χ_W the characters of V, W. Suppose that*

$$\chi_V-\chi_W=\varepsilon_k(\lambda+\rho)\textstyle\sum_{Q\subset P_n}(-1)^{|Q_\lambda|}\chi(\lambda+\langle Q\rangle)$$

for some $\lambda\in L$ such that $\varepsilon(\lambda+\rho)\neq 0$. Then

$$\operatorname{Trace}\pi_V-\operatorname{Trace}\pi_W=(-1)^{|Q_\lambda|}\Theta_{\omega(\lambda+\rho)}$$

where $Q_\lambda=\{\beta\in P_n;(\lambda+\rho,\beta)>0\}$.

Corollary. *For $\lambda\in L$, take such K-modules V, W as $[V]=\oplus[\lambda+\langle Q\rangle]$ where the summation runs over every $Q\subset P_n$ such that $\varepsilon_k(\lambda+\rho)\varepsilon_k(\lambda+\rho_k+\langle Q\rangle)=(-1)^{|Q|}$, and as $[W]=\oplus[\lambda+\langle Q\rangle]$ where the summation runs over every $Q\subset P_n$ such that $\varepsilon_k(\lambda+\rho)\varepsilon_k(\lambda+\rho_k+\langle Q\rangle)=(-1)^{|Q|+1}$. Then for any first order operator D, the formula in Theorem 1 holds. Here, $[V],[W]$ denote the equivalence classes to which V, W belong.*

3. Let $V_{\lambda+\langle Q\rangle}$ be an irreducible K-module belonging to $[\lambda+\langle Q\rangle]$ for $\lambda\in L, Q\subset P_n$, when $\varepsilon_k(\lambda+\rho_k+\langle Q\rangle)\neq 0$, and denote by $w_{\lambda+\langle Q\rangle}$ the unique element of W_G such that $w_{\lambda+\langle Q\rangle}(\lambda+\rho_k+\langle Q\rangle)$ is k^C-dominant. Let Ω be the Casimir operator of G. Then the action of Ω on $L_2(G)$ as a left invariant differential operator defines the action $\nu(\Omega)$ on $L_2(\mathcal{V}_{\lambda+\langle Q\rangle})$ because Ω belongs to the center of the universal enveloping algebra of $\mathfrak{g}^C$. Put $H_\lambda^Q=\{f\in L_2(\mathcal{V}_{\lambda+\langle Q\rangle});\nu(\Omega)f=(\lambda+2\rho,\lambda)f\}$. For $w\in W_G$, we put $A_\lambda(w,Q)=(\rho-\langle Q\rangle,2\langle Q_\lambda\rangle-\langle Q\rangle-\rho)/2+(\rho_k,\rho_n-\langle Q_\lambda\rangle-w(\rho_n-\langle Q_\lambda\rangle))+(\rho,\rho)/2$. We then have the following lemma in a similar way to the one in [6].

Lemma. *If $|(\lambda+\rho,\beta)|>A_\lambda(w_{\lambda+\langle Q\rangle},Q)$ for every $\beta\in P_n$, then $H_\lambda^Q=0$ if $Q\neq Q_\lambda$.*

In [4], we obtained an elliptic complex $\mathcal{V}_\lambda^*$ whose first term is the homogeneous vector bundle associated to an irreducible K-module V_λ with lowest weight $\lambda+2\rho_k$ (the (♯)-complex for λ under an admissible linear order of Δ in terminology of [4]). One can define the square-integrable "cohomology" space $H_2^q(\mathcal{V}_\lambda^*)$ for this elliptic complex. The following proposition is shown by Theorem 1 and the above Lemma.

Proposition. *There exists a non-negative constant a such that the following holds. If $|(\lambda+\rho,\alpha)|>a$ for every $\alpha\in P$, then $H_2^{q_\lambda}(\mathcal{V}_\lambda^*)\neq 0$ and the irreducible unitary representation of G with character $\Theta_{\omega(\lambda+\rho)}$ is realized as a closed subspace of $H_2^{q_\lambda}(\mathcal{V}_\lambda^*)$ for $q_\lambda=|Q_\lambda|$.*

4. For $\Lambda\in L'$, choose a positive root system such as $P=\{\alpha\in\Delta;(\Lambda,\alpha)<0\}$ and fix the linear order on Δ induced by P. Put $\lambda=\Lambda-\rho$. Then $-(\lambda+2\rho_k)$ is $\mathfrak{k}^C$-dominant with respect to this linear order. Let V_λ be the irreducible K-module with lowest weight $\lambda+2\rho_k$, and put $A(w,Q)=(\langle Q\rangle,\langle Q\rangle)/2+(\rho_k,\rho_n-w\rho_n)$ and $b=\max_{w\in W_G,Q\subset P_n}A(w,Q)$. The next theorem follows from Corollary to Theorem 1 and Lemma in **3**.

Theorem 2. *If $|(\lambda+\rho,\beta)|>b$ for every $\beta\in P_n$, then the Hilbert space*

$$\mathfrak{H}_\lambda=\{f\in L_2(\mathcal{V}_\lambda);\nu(\Omega)f=(\lambda+2\rho,\lambda)f\}$$

gives an irreducible unitary representation belonging to the discrete series for G, whose character is $\Theta_{\omega(\lambda+\rho)}$.

Remark. In view of Harish-Chandra's result cited in **1**, we see that "most" of the discrete series are realized in this procedure. This construction is partially a generalization of the method in [9] for the de Sitter group and an answer to the proposal in [6]. Further, when (G,K) is a symmetric pair of hermitian type and that all elements in P_n are totally positive, Theorem 2 is included in Proposition 9.1 in [5].

5. As for a relation with another realization of the discrete series, we shall refer to the one by means of Schmid's operator (see [4], [7]). For $\Lambda\in L'$, we choose P and define λ, V_λ as in **4**. Put $c^\sharp=|\min_{\alpha\in P_k,Q\subset P_n}(\rho_n-\langle Q\rangle,\alpha)|$. Then λ satisfies the condition (♯) in terminology of [4] if $|(\lambda+\rho, \alpha)| \geqslant c^{\#}$ for every $\alpha \in P_k$. Hence for a K-module V_λ^1 whose irreducible components consist of $[\lambda+\beta]$ where β runs over the elements on P_n, we have an elliptic first order operator

$$\mathcal{D}: L_2(\mathcal{V}_\lambda) \rightarrow L_2(\mathcal{V}_\lambda^1)$$

(see [4], [7]). We denote by $\mathcal{H}_\lambda$ the null space of $\mathcal{D}$. Put $c' = |\min_{\alpha \in P_k, \beta \in P_n} (\rho_k - \rho_n - \beta, \alpha)|$. It is then easily seen that the multiplicity of $[\lambda]$ in $\mathcal{H}_\lambda$ is at most one, if $|(\lambda+\rho, \alpha)| > \max(c^{\#}, c')$ for every $\alpha \in P_k$, from Theorem 6.2 in [4] (see also [7]). Taking the unique element $w_0 \in W_G$ such that $w_0 P_k = -P_k$, we put $c'' = \max_{Q \subset P_n} A(w_0, Q)$, and $c = \max(c^{\#}, c', c'')$. Combining the above fact with Theorem 1 and Lemma in 3, one can complete a proof of the following theorem.

Theorem 3*). *If* $|(\lambda+\rho, \alpha)| > c$ *for every* $\alpha \in \Delta$, *then* $\mathcal{H}_\lambda$ *gives an irreducible unitary representation with character* $\Theta_{\omega(\lambda+\rho)}$.

Remark. Under the condition of Theorem 3, we see that $\mathcal{H}_\lambda$ is contained in $\mathfrak{H}_\lambda$. Moreover, we can show that $\mathfrak{H}^\lambda$ is irreducible, which implies that $\mathfrak{H}_\lambda = \mathcal{H}_\lambda$. Therefore, under this condition, the two procedures to realize the discrete series are equivalent.

References

[1] V. Bargmann: Irreducible unitary representations of the Lorentz group. Ann. of Math., **48**, 568–640 (1947).

[2] Harish-Chandra: Representations of semisimple Lie groups. V. Amer. J. Math., **78**, 1–41 (1956).

[3] ——: Discrete series for semisimple Lie groups. II. Acta Math., **116**, 1–111 (1966).

[4] R. Hotta: Elliptic complexes on certain homogeneous spaces. Osaka J. Math., **7**, 117–160 (1970).

[5] M. S. Narasimhan and K. Okamoto: An analogue of the Borel-Weil-Bott theorem for hermitian symmetric pairs of non-compact type. Ann. of Math., **91**, 486–511 (1970).

[6] K. Okamoto: On induced representations. Osaka J. Math., **4**, 85–94 (1967).

[7] W. Schmid: Homogeneous complex manifolds and representations of semisimple Lie groups. Thesis, Proc. Nat. Acad. Sci. U.S.A., **59**, 56–59 (1968).

[8] ——: On a conjecture of Langlands· to appear.

[9] R. Takahashi: Sur les représentations unitaires des groupes de Lorentz généralisés. Bull. Soc. Math. France, **91**, 289–433 (1963).

*) The fact in Theorem 3 was communicated in the letter from Prof. Schmid without a proof.