

257. A Table of Fundamental Units of Purely Cubic Fields

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The following table shows the fundamental unit of the real cubic field $\mathcal{Q}(\sqrt[3]{m})$, for $2 \leq m \leq 250$. In the field $\mathcal{Q}(\sqrt[3]{m})$, there is only one fundamental unit $\varepsilon (> 1)$, and any unit in $\mathcal{Q}(\sqrt[3]{m})$ can be expressed as $\pm \varepsilon^n (n \in \mathbf{Z})$ by Dirichlet theorem.

In 1892, A. A. Markov published the table of a unit of $\mathcal{Q}(\sqrt[3]{m})$, for $2 \leq m \leq 70$. But in his table, two units were not fundamental units ($m = 28, 55$). In 1896, G. F. Voronoi gave an algorithm of finding fundamental units of fields $\mathcal{Q}(\sqrt[3]{m})$. It was based on a generalization of the algorithm of continued fractions. In 1955, K. K. Billebič [1] gave a method of finding a system of fundamental units of any algebraic number field.

In 1900, R. Dedekind [3] calculated ideal class numbers of $\mathcal{Q}(\sqrt[3]{m})$ by analytic method using Markov's table. If the author's table is used, ideal class numbers of $\mathcal{Q}(\sqrt[3]{m})$, $2 \leq m \leq 250$, can be calculated. Recently the author has found an algorithm of calculating the structure of ideal class group of any finite algebraic number field by geometrical method. This method will be published later.

For computing this table, the author used the electronic computer TOSBAC-3300 installed in the Department of Mathematics University of Tokyo. This calculation required five hours computer time.

Table

m	Fundamental unit (> 1), $\alpha = \sqrt[3]{m}$
2	$1 + \alpha + \alpha^2$
3	$4 + 3\alpha + 2\alpha^2$
5	$41 + 24\alpha + 14\alpha^2$
6	$109 + 60\alpha + 33\alpha^2$
7	$4 + 2\alpha + \alpha^2$
10	$(23 + 11\alpha + 5\alpha^2)/3$
11	$89 + 40\alpha + 18\alpha^2$
12	$(110 + 48\alpha + 21\alpha^2)/2$
13	$94 + 40\alpha + 17\alpha^2$
14	$29 + 12\alpha + 5\alpha^2$
15	$5401 + 2190\alpha + 888\alpha^2$
17	$324 + 126\alpha + 49\alpha^2$
19	$(14 + 5\alpha + 2\alpha^2)/3$

Table (Continued)

m	Fundamental unit (>1), $\alpha = \sqrt[m]{m}$
20	$(22 + 8\alpha + 3\alpha^2)/2$
21	$1705 + 618\alpha + 224\alpha^2$
22	$793 + 283\alpha + 101\alpha^2$
23	$2166673601 + 761875860\alpha + 267901370\alpha^2$
26	$9 + 3\alpha + \alpha^2$
28	$(10 + 4\alpha + \alpha^2)/6$
29	$102866541757601689 + 33481749309704842\alpha + 10897883001448120\alpha^2$
30	$811 + 261\alpha + 84\alpha^2$
31	$101209 + 32218\alpha + 10256\alpha^2$
33	$15270674074129 + 4760876269140\alpha + 1484279131362\alpha^2$
34	$334153 + 103146\alpha + 31839\alpha^2$
35	$(278 + 85\alpha + 26\alpha^2)/3$
37	$100 + 30\alpha + 9\alpha^2$
38	$29071 + 8647\alpha + 2572\alpha^2$
39	$529 + 156\alpha + 46\alpha^2$
41	$931197095781897587447729 + 270051748734525954034260\alpha + 78316338533401657636358\alpha^2$
42	$21169 + 6090\alpha + 1752\alpha^2$
43	$49 + 14\alpha + 4\alpha^2$
44	$(8014 + 2270\alpha + 643\alpha^2)/6$
45	$1477441 + 415374\alpha + 116780\alpha^2$
46	$16449049 + 4590798\alpha + 1281255\alpha^2$
47	$562944292769 + 155990973316\alpha + 43224852030\alpha^2$
51	$107846641 + 29081484\alpha + 7841994\alpha^2$
52	$209 + 56\alpha + 15\alpha^2$
53	$113015453598 + 30087022392\alpha + 8009779969\alpha^2$
55	$32947340560201 + 8663621462574\alpha + 2278130361072\alpha^2$
57	$1460968 + 379620\alpha + 98641\alpha^2$
58	$929 + 240\alpha + 62\alpha^2$
59	$21618361237973511050873 + 5553141829215933501576\alpha + 1426444115533632242954\alpha^2$
60	$2161 + 552\alpha + 141\alpha^2$
61	$3905 + 992\alpha + 252\alpha^2$
62	$8929 + 2256\alpha + 570\alpha^2$
63	$16 + 4\alpha + \alpha^2$
65	$16 + 4\alpha + \alpha^2$
66	$9505 + 2352\alpha + 582\alpha^2$
67	$4289 + 1056\alpha + 260\alpha^2$
68	$2449 + 600\alpha + 147\alpha^2$
69	$404886837053487091694212951195653956127452401 + 98715184393700556938337454013404500951638820\alpha + 24067681974543893805323831567684099602695630\alpha^2$
70	$1121 + 272\alpha + 66\alpha^2$
71	$1788355606552816482 + 431884645684316172\alpha + 104299361097095425\alpha^2$
73	$99928 + 23910\alpha + 5721\alpha^2$
74	$1025641 + 244297\alpha + 58189\alpha^2$
76	$305 + 72\alpha + 17\alpha^2$
77	$3752144035327073 + 881960753382348\alpha + 207309411148166\alpha^2$
78	$9818137169569 + 2297898779424\alpha + 537814730970\alpha^2$
79	$370454 + 86336\alpha + 20121\alpha^2$
82	$(7644966923903 + 1759696053245\alpha + 405041673905\alpha^2)/3$

Table (Continued)

m	Fundamental unit (>1), $\alpha = \sqrt[m]{m}$
83	$7512137677415067474067653515153 + 1722149465980481177969345011132\alpha + 394800908946784666615761008466\alpha^2$
84	$(332642 + 75954\alpha + 17343\alpha^2)/2$
85	$658895013725266441 + 149856842965183254\alpha + 34082931143344968\alpha^2$
86	$565 + 128\alpha + 29\alpha^2$
87	$86965911738730294017560094302901018001 + 19626489977891548853457921605649509148\alpha + 442931145262436057399605984156302924$
89	$6\alpha^2(1124105870 + 251773807\alpha + 56391530\alpha^2)/3$
90	$58321 + 13014\alpha + 2904\alpha^2$
91	$81 + 18\alpha + 4\alpha^2$
92	$(214312438 + 47473520\alpha + 10516119\alpha^2)/2$
93	$90139756288 + 19895524677\alpha + 4391313206\alpha^2$
94	$961302655531 + 211422331357\alpha + 46498781564\alpha^2$
95	$3576393871601 + 783797980668\alpha + 171776179178\alpha^2$
97	$1040068758302891753134 + 226362669236408190113\alpha + 49266029399288296684\alpha^2$
99	$2370504989476787061889 + 512423622402913328862\alpha + 110768789756683516940\alpha^2$
101	$378735919297636937009 + 81325992283885669708\alpha + 17463136407088330740\alpha^2$
102	$86538093769 + 18521405235\alpha + 3964062957\alpha^2$
103	$2280708165087158401 + 486546077660832382\alpha + 103795430432939276\alpha^2$
105	$840883681 + 178240404\alpha + 37781256\alpha^2$
106	$170689681 + 36066609\alpha + 7620849\alpha^2$
107	$(113247787855207200198554 + 23854398394735212748450\alpha + 5024666119768108586783\alpha^2)/3$
109	$102488185 + 21455154\alpha + 4491480\alpha^2$
110	$551 + 115\alpha + 24\alpha^2$
111	$2938440619017044137912 + 611424155687278152855\alpha + 127223771594525638322\alpha^2$
113	$4817022058571097089 + 996366584271774000\alpha + 206091306658430912\alpha^2$
114	$4133238949 + 852423792\alpha + 175800705\alpha^2$
115	$303330146589001 + 62375823865520\alpha + 12826761357730\alpha^2$
116	$(16628482262 + 3409572648\alpha + 699112851\alpha^2)/2$
117	$128794 + 26333\alpha + 5384\alpha^2$
118	$(698690155699793 + 142448305839101\alpha + 29042229478826\alpha^2)/3$
119	$2321770662645197401 + 472030794407215522\alpha + 95966873237538304\alpha^2$
122	$15251 + 3075\alpha + 620\alpha^2$
123	$321196922237148213101401 + 64585695098239197741462\alpha + 12986774537780061163896\alpha^2$
124	$25 + 5\alpha + \alpha^2$
126	$25 + 5\alpha + \alpha^2$
127	$217409401369 + 43252420170\alpha + 8604834192\alpha^2$
129	$4802014888 + 950371926\alpha + 188089129\alpha^2$
130	$5851 + 115\alpha + 228\alpha^2$
131	$691270935982115863484889755793001 + 136110365152124040407110318493750\alpha + 26799957205670565681905244500960\alpha^2$
132	$90327013921 + 17740247571\alpha + 3484188951\alpha^2$
133	$4593394 + 899877\alpha + 176292\alpha^2$
134	$(62652683111 + 12243476209\alpha + 2392598405\alpha^2)/3$
137	$9780527118070486621555332662035098702880452426329942572801 + 1897239126707243962422173536010141514973994562427109790010\alpha + 368028865975782124442749645644485095299191945728688477740\alpha^2$
138	$7525246833430849 + 1456222445822709\alpha + 281795914294425\alpha^2$
139	$98213793808004 + 18959820464226\alpha + 3660125305193\alpha^2$
140	$701 + 135\alpha + 26\alpha^2$

Table (Continued)

m	Fundamental unit (>1), $\alpha = \sqrt[m]{m}$
141	$90941859649 + 17472596988\alpha + 3356998050\alpha^2$
142	$552019321 + 105809541\alpha + 20281281\alpha^2$
143	$(7305401975 + 1397007610\alpha + 267148922\alpha^2)/3$
145	$4601120912488201407747405990121 + 875805454131464787106374958632\alpha + 166706158798079221119438870726\alpha^2$
146	$658031496221030890375 + 124967110857372704197\alpha + 23732570379569259472\alpha^2$
148	$1878269 + 355089\alpha + 67130\alpha^2$
149	$859699668151966502259032173489 + 162162837995150600262255107252\alpha + 30588340324906413785889988460\alpha^2$
150	$(1355 + 255\alpha + 48\alpha^2)/5$
151	$58886554777961 + 11058354219854\alpha + 2076657371328\alpha^2$
153	$1888 + 353\alpha + 66\alpha^2$
154	$(2155235855 + 402088109\alpha + 75014921\alpha^2)/3$
155	$807243088561096721 + 150277433472159724\alpha + 27975844365832758\alpha^2$
156	$(3789756830 + 703995384\alpha + 130776069\alpha^2)/2$
157	$4012030799617318079361745 + 743699873398009028217598\alpha + 137857740709510585418372\alpha^2$
158	$89455153351649 + 16547015316624\alpha + 3060793097210\alpha^2$
159	$830240612335926573133077433 + 153251569942463356234249896\alpha + 28288237579405435437820554\alpha^2$
161	$39275546844846 + 7219607961969\alpha + 1327104096868\alpha^2$
163	$512936998354 + 93900554724\alpha + 17189858025\alpha^2$
164	$162361 + 29662\alpha + 5419\alpha^2$
165	$130681 + 23826\alpha + 4344\alpha^2$
166	$1767569 + 321618\alpha + 58520\alpha^2$
167	$2070166880101044205370118762758521169120826289178053244249389075109893279095936$ $78542437176000329 +$ $3759233378430387052058014955399294460976175786237429132766935800446953722774928$ $4245814885920536\alpha +$ $6826441540391252694667966480938649885916157400845126642231195944528075516296200$ $936950484198110\alpha^2$
170	$175601374201 + 31698954351\alpha + 5722185897\alpha^2$
171	$6100 + 1099\alpha + 198\alpha^2$
172	$(117310 + 21094\alpha + 3793\alpha^2)/6$
173	$1298828639084088159278484528360730410533356984157704652806743334753 +$ $233096895019168539395020385397325352385828994785662845633023003262\alpha +$ $41833203266824178774723154719985171566527663453915879687183651996\alpha^2$
174	$235444969 + 42173502\alpha + 7554225\alpha^2$
175	$(378005 + 67580\alpha + 12082\alpha^2)/5$
177	$1529999819870129383038273618656214656864166186711884183651770943819074875333937$ $7449993 +$ $2725002829532353198471015944634228563892804306074581570673840009815879882246402$ $575848\alpha +$ $4853360323656534864543756607695913090288163008411354585576914770631365148042008$ $51406\alpha^2$
178	$4218014114104368961 + 749839005206339784\alpha + 133299348584142966\alpha^2$
179	$1565882142419862384470856245195930911194 +$ $277848502886388600942247003648177715976\alpha +$ $49301150108848866524199807763681342081\alpha^2$
181	$(16966724329842756278 + 2999428257712544915\alpha + 530247895720239530\alpha^2)/3$
182	$289 + 51\alpha + 9\alpha^2$
183	$4233504031039334302315890452620079552467606161001 +$ $745675054088764429586069727215344121157283327710\alpha +$ $131340677182200506036311403064866045202757322400\alpha^2$

Table (Continued)

m	Fundamental unit (>1), $\alpha = \sqrt[3]{m}$
185	$119999880001 + 21059928980\alpha + 3696008768\alpha^2$
186	$2152107793 + 377015934\alpha + 66047349\alpha^2$
187	$317918976327167993137411436881 + 55595023208207230924300971516\alpha + 9721994708300693348196450546\alpha^2$
188	$(829001404477710769250 + 144711368281529982102\alpha + 25260970604876278431\alpha^2)/2$
190	$(72361501175463503 + 12587023246682465\alpha + 2189467488082553\alpha^2)/3$
191	$140093388509230527707238349852308120053666369 + 24326139010308574455662367161624244972108700\alpha + 4224046869348630600384506538269215409189568\alpha^2$
193	$310748091322658101526700649 + 53771980623158197557987634\alpha + 9304726178141044079043680\alpha^2$
194	$6813435220110859864449929 + 1176970437212281567653930\alpha + 203312920035237376952183\alpha^2$
195	$103412962067391156961 + 17833233965348046792\alpha + 3075284058255639852\alpha^2$
197	$326076733722381241980431716564600534043415320035464220797168 + 56039949683817422135451445856527678879031271436671660719670\alpha + 9631094879767045845289157130803107138502912631297689022961\alpha^2$
198	$1189 + 204\alpha + 35\alpha^2$
199	$(29927 + 5126\alpha + 878\alpha^2)/3$
201	$282914974686875109517742143969 + 48297418257891394288011264918\alpha + 8245023484385805591032095740\alpha^2$
202	$12169335834253294717914317000263995001 + 2074036618636606246086611794768835095\alpha + 353480909232341299561794101364036425\alpha^2$
203	$514062295828 + 87468243542\alpha + 14882814185\alpha^2$
204	$(5510 + 936\alpha + 159\alpha^2)/2$
205	$83465288398231324201 + 14155371705895912352\alpha + 2400693174101877298\alpha^2$
206	$(5808269975 + 983462311\alpha + 166520861\alpha^2)/3$
207	$4969 + 840\alpha + 142\alpha^2$
209	$377290 + 63576\alpha + 10713\alpha^2$
210	$11341 + 1908\alpha + 321\alpha^2$
211	$28578321115120724692526 + 4800383047391675701035\alpha + 806334189781814499440\alpha^2$
212	$(51518 + 8640\alpha + 1449\alpha^2)/2$
213	$46009 + 7704\alpha + 1290\alpha^2$
214	$104005 + 17388\alpha + 2907\alpha^2$
215	$36 + 6\alpha + \alpha^2$
217	$36 + 6\alpha + \alpha^2$
218	$105949 + 17604\alpha + 2925\alpha^2$
219	$47305 + 7848\alpha + 1302\alpha^2$
220	$(53462 + 8856\alpha + 1467\alpha^2)/2$
221	$3581449162473160275053095200699362120441 + 592372241916916996898572116020362358424\alpha + 97978459856556500620606683613369693274\alpha^2$
222	$11989 + 1980\alpha + 327\alpha^2$
223	$290403613332741389389409 + 47888775020014619589782\alpha + 7897060048939189665372\alpha^2$
226	$(194968979780302075519284111242477571357023 + 32008307167762934859643910480492725424009\alpha + 5254844790696153647342349343725673704617\alpha^2)/3$
227	$1639936749349455607139426288533822861960291966689461391238127917704745971968901 + 4614532657389414481 + 2688345891142163539585994670105740842813821097315919941499430585054066135174822 + 859175691084707398\alpha + 4407001448859477963339877269913681898590204182765139561536906160346382573213792 + 23782346642118676\alpha^2$

Table (Continued)

m	Fundamental unit (>1), $\alpha = \sqrt[m]{m}$
228	$(6158 + 1008\alpha + 165\alpha^2)/2$
229	$45665 + 7464\alpha + 1220\alpha^2$
230	$23809996981 + 3886124664\alpha + 634269921\alpha^2$
231	$44914 + 7320\alpha + 1193\alpha^2$
233	$(386519830942502 + 62813521316866\alpha + 10207855184567\alpha^2)/3$
234	$1405 + 228\alpha + 37\alpha^2$
235	$(229338383 + 37163858\alpha + 6022334\alpha^2)/3$
236	$13901581 + 2249539\alpha + 364018\alpha^2$
237	$16256982906342583000470053231327892613176118654 +$ $2626982423778989162357847356780028710739354756\alpha +$ $424496765150151362812472543258871517156352249\alpha^2$
238	$10624321 + 1714387\alpha + 276641\alpha^2$
239	$1297210666070701695128424020205447169311981562637931767629831831847637335154224$ $6445449784249440932371355990207548750021841241680343074292776472564418698366859$ $864906567294925737472124385185 +$ $2090312466171118485744217392622240864356232614474027745711317658247883084085566$ $3855314750825580355585325403316298297126656439845907573193746257843228824316927$ $96407651741158025184644414948\alpha +$ $3368308880365186741611096692335036277799365038863631233365617782384319138983248$ $0710694770736125720916800646155415445039031513414259559821610417463203990814605$ $5728760523786803598551081222\alpha^2$
241	$1010320852597992917321596786624 + 162350502016341453001703057210\alpha +$ $26088430657627754711967317353\alpha^2$
244	$(1035104946 + 165648602\alpha + 26508867\alpha^2)/2$
246	$1278056793936886730798199478071713668598271054581873251 +$ $203972577680300684983807733369748320549630887754141625\alpha +$ $32553179673172516899058456925099480370680361875515440\alpha^2$
247	$3106162477570 + 495060775756\alpha + 78902882081\alpha^2$
249	$9115694171340598924880665761574035823569796182762397129 +$ $1448960781078626614634391426899068579521519507254051502\alpha +$ $230315684756591912246482085511242733348857722291840680\alpha^2$

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