255. *The Decomposition of $L^2(\Gamma\backslash SL(2,R))$ and Teichmüller Spaces*

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0. Let Γ be a discrete subgroup of $G=SL(2,\boldsymbol{R})/\{\pm e\}$ and let χ be a finite dimensional representation of Γ by unitary matrices. We assume that $\Gamma\backslash G$ is compact.

It is well known that the unitary representation $U^{\Gamma,\chi}$ of G induced from χ can be decomposed into the discrete direct sum $\sum_i \oplus U_i$ of irreducible unitary representations U_i of G. We call the set $\{U_i\}$ the spectra of $U^{\Gamma,\chi}$.

The problem we want to study is the following[1]:

"*How do the spectra of $U^{\Gamma,\chi}$ behave when Γ varies?*"

Detailed proofs will appear elsewhere.

1. Let $H=\{z=x+iy\,;\,y>0\}$ be the complex upper half plane. G acts on H transitively by

$$g(z)=\frac{az+b}{cz+d}$$

for z in H and $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in G.

The G-invariant metric on H is

$$ds^2=\frac{dx^2+dy^2}{y^2}.$$

Hence, the G-invariant measure on H is

$$dm(z)=\frac{dxdy}{y^2}$$

and the ring of G-invariant differential operators on H is generated by

$$(1)\qquad \Delta=-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

2. Let V be the representation space of χ. Consider the complex vector space $\mathcal{H}(\Gamma,\chi)$ of all V-valued functions F on H wich satisfy the following conditions:

(i) F is (componentwisely) measurable;

(ii) $F(AZ)=\chi(A)F(z)$ for all z in H and A in Γ;

(iii) $\int_{\mathcal{F}} {}^tF(z)\overline{F(z)}dm(z)<\infty$ where $\mathcal{F}$ is a measurable fundamental

1) Note that some problems of the similar nature were also discussed by J. M. G. Fell [3].

domain of Γ in H.
Introducing an innerproduct.

$$(F_1, F_2)=\int_{\mathcal{F}} {}^tF_1(z)\overline{F_2(z)}dm(z),$$

we can consider $\mathcal{H}(\Gamma, \chi)$ as a Hilbert space.

Now we consider an eigenvalue problem:

$$\text{(2)} \qquad \Delta F=\lambda F \qquad (F\in \mathcal{H}(\Gamma, \chi))$$

where Δ is the differential operator given by (1).

Definition 1. We call the eigenvalue problem (2) *the (Γ, χ)-eigenvalue problem* or *the (Γ, χ)-problem.*

The following theorem is well known.

Theorem.

(i) *$\mathcal{H}(\Gamma, \chi)$ has a complete orthonormal system $\{F_j;\, j=1,2,3\cdots\}$ which consists of eigenvectors of the (Γ, χ)-eigenvalue problem.*

(ii) *The eigenvalues λ_j $(j=1,2,3\cdots)$ of the (Γ, χ)-problem are all real and nonnegative and of finite multiplicity. Moreover, they have no finite point of accumulation on the real line.*

For the correspondence between the spectra of the (Γ, χ)-eigenvalue problem and the spectra of the unitary representation $U^{\Gamma,\chi}$, the reader should refer to the first chapter of [4].

3. In the following, we shall assume that Γ has no elliptic element. Then, as is well known, any closed Riemann surface of genus $g>1$ can be represented as a quotient space $\Gamma\backslash H$ by some such Γ.

Definition 2. We say that Γ is *a Fuchsian group of genus* $g(>1)$, if

(i) Γ is a discrete subgroup of G, such that $\Gamma\backslash H$ is compact,

(ii) Γ has no elliptic element,

(iii) as a closed Riemann surface, the genus of $\Gamma\backslash H$ is g.

It is also well known that two Fuchsian groups of the same genus are isomorphic to each other (as abstract groups).

Now, we can give the definition of Teichmüller spaces.

Definition 3. Fix a Fuchsian group Γ_0 of genus g. Consider the set $\tilde{T}(\Gamma_0)$ of pairs (Γ, θ) where Γ is a Fuchsian group of genus g and θ is an isomorphism from Γ_0 onto Γ. We define an equivalence relation $\sim$ in $\tilde{T}(\Gamma_0)$ by setting $(\Gamma_1, \theta_1)\sim(\Gamma_2, \theta_2)$ if and only if there exists an element B of G such that $\theta_2(A)B=B\theta_1(A)$ for all A in Γ_0.

The set of all equivalence classes under this relation is called *the Teichmüller space* $T(\Gamma_0)$. We denote an element of $T(\Gamma_0)$, which is the class containing (Γ, θ), by $\langle\Gamma, \theta\rangle$.

$T(\Gamma_0)$ has a natural structure of a $(3g-3)$-dimensional complex manifold ($g=$genus of Γ_0). It also has a structure of a Kähler manifold (see [1] and [2]).

4. Let Γ_0 be a fixed Fuchsian group of genus g and let χ be its fixed finite dimensional representation by unitary matrices.

With each element (Γ, θ) in $\tilde{T}(\Gamma_0)$, we associate the (Γ, χ_θ)-eigenvalue problem, where χ_θ is a unitary matrix-representation of Γ defined by

$$\chi_\theta(A)=\chi(\theta^{-1}A) \qquad (A \in \Gamma).$$

Then, the correspondence:

$\langle\Gamma, \theta\rangle \mapsto$ the spectra of the (Γ, χ_θ)-problem

defined on $T(\Gamma_0)$ is "well defined".

5. Now we can state the main theorems of this paper.

Theorem A. *For every real analytic curve $C=C(t)$ (t: a real parameter) in $T(\Gamma_0)$, there exists a series $\Lambda^{(j)}=\Lambda^{(j)}(t)$ of real analytic functions in t whose values at each point represent all the eigenvalues of the associated problem.*

Theorem A′. *Given a point P of $T(\Gamma_0)$ and a simple eigenvalue λ_0 of the eigenvalue problem associated with P, there exists a neighbourhood W of P in $T(\Gamma_0)$ and a real analytic function Λ on W such that $\Lambda(P)=\lambda_0$ and, at each point in W, the value of Λ represents one of the eigenvalues of the associated problem.*

To state the next theorem, we must recall some facts in the theory of Teichmüller spaces ([2]).

Let $P=\langle\Gamma, \theta\rangle$ be a point in $T(\Gamma_0)$. The (real) tangent space $\mathcal{T}_P$ of $T(\Gamma_0)$ at P can be identified with the space of functions on $H: \Phi(\Gamma) =\{y^2\overline{\varphi(z)}\,;\ \varphi$ is a Γ-automorphic form of weight 2$\}$ considered as a real vector space.

Theorem B. *Let the notations be as in Theorem A. Given a point $P=C(t_0)=\langle\Gamma_{(t_0)}, \theta_{(t_0)}\rangle$ on the curve C and one of the eigenvalues $\lambda_0 =\Lambda^{(j_1)}(t_0)=\cdots=\Lambda^{(j_m)}(t_0)$ of the $(\Gamma_{(t_0)}, \chi_{\theta(t_0)})$-problem of multiplicity m.*

Then,

$$(3)\qquad \begin{aligned} &\frac{1}{m}\sum_{i=1}^{m}\left[\frac{d}{dt}\Lambda^{(j_i)}(t)\right]_{t=t_0} \\ &=\frac{1}{m}\sum_{i=1}^{m}\left\{-8\,\mathrm{Re.}\int_{\mathcal{F}} {}^t\!\left(\frac{\partial F^{(j_i)}}{\partial \bar{z}}\right)\left(\frac{\partial \bar{F}^{(j_i)}}{\partial \bar{z}}\right)\overline{\nu(z)}\,dxdy\right\}\end{aligned}$$

where ν is the element of $\mathcal{T}_P=\Phi(\Gamma_{(t_0)})$ tangential to the curve C at P and $\{F^{(j_i)}\ (i=1,2,\cdots,m)\}$ is an orthonormal basis of the eigenspace belonging to the eigenvalue λ_0.

Remark. As can be easily shown, $y^2\,{}^t(\partial F^{(j_i)}/\partial\bar{z})(\partial\bar{F}^{(j_i)}/\partial\bar{z})$ $(i=1,2,\cdots,m)$ are Beltrami differentials with respect to $\Gamma(t_0)$ ([1], [2]). Hence the right member of (3) can be rewritten using the innerproduct of $\mathcal{T}_P$ given in [2].

References

[1] L. V. Ahlfors: Lectures on Quasiconformal Mappings. Van Nostrand, Toronto (1966).

[2] ——: Some remarks on Teichmüller's spaces of Riemann surfaces. Ann. Math., **74**, 171–191 (1961).

[3] J. M. G. Fell: Weakcontainment and induced representations of groups. II. Trans. Amer. Math. Soc., **110**, 424–447 (1964).

[4] I. M. Gel'fand, M. I. Graev, and I. I. Pyatetsky-Shapiro: Theory of Representations and Automorphic Functions (in Russian). Hauka, Moskow (1966).