253. *Stable Properties of Gaussian Flows*

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1. It is important to study the stability of dynamical systems as a generalization of mixing property. The strong and the weak stabilities for an automorphism on a probability space were studied by A. Maitra [3] and by S. Natarajan and K. Viswanath [4] (cf. Renyi [5]).

In this paper we shall study the stabilities (mixing property) of a Gaussian flow (flow of the Brownian motion) together with skew product flow of it and a measurable flow with pure point spectrum. As will be seen later, the stabilities coincide with the corresponding mixing properties on each ergodic part of a given dynamical system. Anzai's method in [1] and [2] of skew product dynamical systems is very useful to construct some kinds of models in ergodic theory. In fact we shall be able to give some characteristic properties of a Gaussian process and a Brownian motion by using such a method in § 3 and § 4.

2. Let $(\Omega, \mathscr{B}, m)$ be a probability measure space on which a measurable flow $\{T_t\}$ is given and $\{U_t\}$ denote the one parameter group of unitary operators induced by $\{T_t\}$.

Definition 1. A flow $\{T_t\}$ is said to be *weakly stable* if there exists a constant $C(f, g)$ such that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |(U_t f, g) - C(f, g)|\, dt = 0 \tag{1}$$

holds for arbitrary bounded measurable functions f and g.

Definition 2. A flow $\{T_t\}$ is called *strongly stable* if there exists a constant $C(f, g)$ such that

$$\lim_{T\to\infty} (U_t f, g) = C(f, g) \tag{2}$$

holds for arbitrary bounded measurable functions f and g.

Definition 3. Let $(f_0, f_1, \cdots, f_r)$ be an arbitrary $(r+1)$-tuple of bounded functions of $L^2(\Omega)$ and $(t_0^n, t_1^n, \cdots, t_r^n)$ be an arbitrary $(r+1)$-tuple of real numbers satisfying the condition:

$$t_0^n < \cdots < t_r^n \text{ and } \lim_{n\to\infty} \min_{1\leq j\leq r} (t_j^n - t_{j-1}^n) = \infty. \tag{3}$$

We call $\{T_t\}$ an *r-order stable* flow if there exists a constant $C(f_0, \cdots, f_r)$ such that

$$\lim \left(\prod_{j=0}^{r} U_{t_j^n} f_j, 1 \right) = C(f_0, \cdots, f_r). \tag{4}$$

If $\{T_t\}$ is r-order stable for any positive integer r, the flow $\{T_t\}$ is said to be *all order stable.*

3. Let $\{x(t, \omega), -\infty<t<\infty\}$ be a real measurable stationary Gaussian process on a probability measure space $(\Omega, \mathcal{B}, m)$. A stationary Gaussian process is completely determined by giving the mean $E\{x(t)\}$ and the covariance $\rho(t)=E\{x(t)x(o)\}$. A Gaussian flow $\{T_t\}$ induced by the process is given as follows: $x(s, T_t\omega)=x(s+t, \omega)$ for all t and s. We may assume without loss of generality the mean of the process is zero: $E\{x(t)\}=0$ for all t. Let $\{S_t\}$ be a measurable flow on another probability measure space (Y, Σ, μ) with pure point spectrum and let us define a skew product flow $\{Z_t\}$ of the flows $\{T_t\}$ and $\{S_t\}$ as follows:

$$Z_t(\omega, y)=(T_t\omega, S_{x(t,\omega)-x(0,\omega)}y), \quad -\infty<t<\infty. \tag{5}$$

Using the fundamental arguments for Gaussian distributions we can prove the following theorem.

Theorem 1. *For a Gaussian flow $\{T_t\}$, the following statements are pairwise equivalent:*

(i) *$\{T_t\}$ is strongly stable (i.e. 1-rder stable),*

(ii) *$\{T_t\}$ is all order stable,*

(iii) *It holds that* $\lim_{t\to\infty}\rho(t)=C$

for some constant C (note that the constant C is not necessarily equal to zero).

Corollary 2 ([6]). *Let $\{T_t\}$ be a Gaussian flow. Then the following statements are pairwise equivalent:*

(i) *$\{T_t\}$ is strongly mixing (i.e. 1-order mixing),*

(ii) *$\{T_t\}$ is all order mixing,*

(iii) *It holds that* $\lim_{t\to\infty}\rho(t)=0$.

Theorem 3. *Let $\{T_t\}$ be a strongly stable Gaussian flow. Then the skew product flow $\{Z_t\}$ of the flows $\{T_t\}$ and $\{S_t\}$ given by* (5) *is all order stable.*

Proof. In order to prove the theorem, it is sufficient to show that the flow $\{Z_t\}$ is r-order stable for any positive integer r. Let $(f_0, \cdots f_r)$ be an arbitrary $(r+1)$-tuple of bounded measurable functions on $\Omega\times Y$ and $(t_0^n, \cdots, t_r^n)$ be an arbitrary $(r+1)$-tuple of real numbers satisfying the condition (3).

Case I. Suppose $f_j(\omega, y)$, $o\leqq j\leqq r$, are functions of the forms:

$$\varphi_j(\omega)=F_j(x(t_{j1}, \omega), \cdots, x(t_{jk_j}, \omega)), \tag{6}$$

$$f_j(\omega, y)=\varphi_j(\omega)\psi_j(y), \qquad o\leqq j\leqq r, \tag{7}$$

where $k_j, t_{j1}, \cdots, t_{jk_j}, o\leqq j\leqq r$, are arbitrary, $F_j(u_1, \cdots, u_{k_j}), o\leqq j\leqq r$, are bounded continuous functions and $\psi_j(y)$, $o\leqq j\leqq r$, are proper functions of $\{S_t\}$. Then the left hand side of (4) is equal to

$$\left(\prod_{j=1}^{r} \psi_j, \bar{\psi}_0\right) \lim_{n\to\infty} \int \left\{ \prod_{j=1}^{r} F_j(x(t_{j1}+t_j^n-t_0^n, \omega), \cdots, x(t_{jk_j}+t_j^n-t_0^n, \omega)) \right.$$
$$\left. \times \exp i\lambda_j(x(t_j^n-t_0^n, \omega)-x(0, \omega)) \right\} \overline{F_0(x(t_{01}, \omega \cdots, x(_{0k_0}, \omega))} dm$$

where $\lambda_j, 1 \leqq j \leqq r$, are proper values of $\{S_t\}$. Next we consider the following functions

$$G_0(x(t_{01}, \omega), \cdots, x(t_{0k_0}, \omega) = \overline{F_0(x(t_{01}, \omega) \cdots, x(t_{0k_0}, \omega))},$$
$$G_j(x(0, \omega), x(t_j^n-t_0^n, \omega), x(t_{j1}+t_j^n-t_0^n, \omega), \cdots, x(t_{jk_j}+t_j^n-t_0^n, \omega))$$
$$= F_j(x(t_{j1}+t_j^n-t_0^n, \omega), \cdots, x(t_{jk_j}+t_j^n-t_0^n, \omega))$$
$$\times \exp i\lambda_j(x(t_j^n-t_0^n, \omega)-x(0, \omega)), \qquad 1 \leqq j \leqq r.$$

The $k_0+k_1+\cdots+k_r+2r$ dimensional random vector

$$\begin{aligned} &(x(t_{01}, \omega), \cdots, x(_{0k_0}, \omega), x(0, \omega), x(t_1^n-t_0^n, \omega), x(t_{11}+t_1^n-t_0^n, \omega) \\ (8) \quad &\cdots, x(t_{1k_1}+t_1^n-t_0^n, \omega), \cdots, x(0, \omega), x(t_r^n-t_0^n, \omega), x(t_{r1}+t_r^n-t_0^n, \omega), \\ &\cdots, x(t_{rk_r}+t_r^n-t_0^n, \omega)) \end{aligned}$$

is Gaussian with the mean vector 0 and the covariance matrix:

$$N_n = \begin{pmatrix} N_{00}(n), N_{01}(n), \cdots, N_{0r}(n) \\ \cdots\cdots\cdots\cdots \\ N_{r0}(n), N_{r1}(n), \cdots, N_{rr}(n) \end{pmatrix}$$

where $N_{00}(n)$ is a $k_0 \times k_0$-matrix with (p, q)-element:

$$(9) \qquad E\{x(t_{0p})x(t_{0q})\} = \rho(t_{0q}-t_{0p})$$

and for $(i, j) \neq (0, 0), N_{ij}(n)$ is a $(k_i+2) \times (k_j+2)$-matrix with (p, q)-element:

$$(10) \quad \begin{aligned} &E\{x(0)x(0)\} = \rho(0) && \text{if } p=q=1 \\ &E\{x(t_i^n-t_0^n)x(t_j^n-t_0^n)\} = \rho(t_j^n-t_i^n) && \text{if } p=q=2, \\ &E\{x(0)x(t_j^n-t_0^n)\} = \rho(t_j^n-t_0^n) && \text{if } p=1 \text{ and } q=2 \\ &E\{x(0)x(t_{jq-2}+t_j^n-t_0^n)\} = \rho(t_{jq-2}+t_j^n-t_0^n) && \text{if } p=1 \text{ and } q>2, \\ &E\{x(t_i^n-t_0^n)x(t_{jq-2}+t_j^n-t_0^n)\} = \rho(t_{jq-2}+t_j^n-t_i^n) && \text{if } p=2 \text{ and } q>2, \\ &E\{x(t_{ip-2}+t_i^n-t_0^n)x(t_{jq-2}+t_j^n-t_0^n)\} = \rho(t_{jq-2}-t_{ip-2}+t_j^n-t_i^n) \\ & && \text{if } p, q>2. \end{aligned}$$

By virtue of Theorem 1, (9) and (10) we have

$$(11) \qquad \lim_{n\to\infty} N_n = \begin{pmatrix} N_{00}, N, \cdots, N \\ \cdot \quad \cdot \quad \cdot \\ N, \quad \cdots, \quad N_{rr} \end{pmatrix} = \tilde{N}$$

where $\lim_{n\to\infty} N_{00}(n) = N_{00}$,

$$\lim_{n\to\infty} N_{ii}(n) = \begin{pmatrix} \rho(0), & C, & C, & C, & \cdots, & C \\ C, & \rho(0), & \rho(t_{i1}), & \rho(t_{i2}), & \cdots, & \rho(t_{ik_i}) \\ C, & \rho(t_{i1}), & \rho(0), & \rho(t_{i2}-t_{i1}), & & \rho(t_{ik_i}-t_{i1}) \\ & & \cdot & & \cdot & \cdot \\ C, & & & & & \rho(0) \end{pmatrix} = N_{ii}$$

and

$$\lim_{n\to\infty} N_{ij}(n)=\begin{pmatrix} \rho(0), C, \cdots, C \\ C, \quad C, & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ C, & & & C \end{pmatrix}=N \qquad \text{for } i\neq j.$$

The relation (11) means that the distribution of the random vector (8) converges the Gaussian distribution of the mean vector 0 and the covariance matrix N. Thus the relation (4) follows. Now denote by $\mathcal{L}$ the set of all finite linear combinations of the functions of the form (6). Then it is easily verified that the equality (4) holds for arbitrary functions $f_j, 0\leqq j\leqq r$, of the form (7) in which $\varphi_j(\omega), 0\leqq j\leqq r$, are in $\mathcal{L}$.

Case II. Let $f_0, \cdots, f_r$ be arbitrary bounded measurable functions on $\Omega\times Y$. Noticing that the family $\mathfrak{A}$ of all proper functions of $\{S_t\}$ is a complete orthogonal system of $L^2(Y)$ and $\mathcal{L}$ is dense in $L^2(\Omega)$, for any positive number ε, there exist functions $g_{jp}, 0\leqq j\leqq r$, of the forms:

$$g_{jp}(\omega, y)=\sum_{k=1}^{n_j^{(p)}} \varphi_{jk}^{(p)}(\omega)\cdot\psi_{jk}^{(p)}(y),$$

$$\varphi_{jk}^{(p)}\in\mathcal{L}, \quad \psi_{jp}^{(p)}\in\mathfrak{A}, \quad 1\leqq k\leqq n_j^{(p)}, \quad 0\leqq j\leqq r, \quad p\geqq 1$$

such that

(12) $$\|f_j-g_{jp}\|<\varepsilon, \quad 0\leqq j\leqq r, \quad p\to\infty.$$

In this case we see that the sequence of constants $C_p(g_{0p}, \cdots, g_{rp})$, $p\geqq 1$, corresponding to the functions $g_{0p}, \cdots, g_{rp}$, $p\geqq 1$, is a Cauchy sequence. So we take

(13) $$C(f_0, \cdots, f_r)=\lim_{p\to\infty} C_p(g_{0p}, \cdots, g_{rp})$$

and choose a positive number M such that

(14) $$|f_j|\leqq M, \quad |g_{jp}|\leqq M, \quad 0\leqq j\leqq r, \quad p\geqq 1.$$

Then from (12), (13) and (14) we obtain

$$\left|\left(\prod_{j=0}^{r} U_{t_j^n} f_j, 1\right)-C(f_0, \cdots, f_r)\right|\leqq\{(r+1)\cdot M^r+2\}\cdot\varepsilon, \quad n\to\infty, \quad p\to\infty.$$

Therefore the proof of the theorem is completed.

Remark. Although a Gaussian flow $\{T_t\}$ is a Kolmogorov flow and the flow $\{S_t\}$ is ergodic, the skew product flow $\{Z_t\}$ of $\{T_t\}$ and $\{S_t\}$ defined by (5) is not ergodic. To show this, note that $x(t,\omega)$ alone has meaning in our case and choose a function $h(\omega, y)=e^{i\lambda x(0,\omega)}\overline{g_\lambda(y)}$, where g_λ is a proper function corresponding to the proper value λ of $\{S_t\}$. Although $h(\omega, y)$ in $L^2(\Omega\times Y)$ is not a constant, it is invariant under the flow $\{Z_t\}$. This contradiction implies that the flow $\{Z_t\}$ is not ergodic. In particular, our assertion is true if the flow $\{S_t\}$ is not weakly mixing.

4. Now let us consider a Brownian motion $\{x(t,\omega), -\infty<t<\infty\}$ on $(\Omega, \mathcal{B}, m)$ and a flow $\{T_t\}$ of the Brownian motion which is given by

$$\Delta x(I, T_t\omega)=\Delta x(I+t, \omega) \qquad -\infty<t<\infty,$$

where $\Delta x(I,\omega)=x(b,\omega)-x(a,\omega)$ for $I=[a, b)$. It is easy to show that

$\{T_t\}$ is a Kolmogorov flow, and thus $\{T_t\}$ is all order mixing. Let $\{S_t\}$ be an ergodic measurable flow on (Y, Σ, μ) with pure point spectrum and consider the skew product flow $\{Z_t\}$ of $\{T_t\}$ and $\{S_t\}$ given by (5). It is to be noticed that $x(t, \omega)$ alone has no meaning.

Theorem 4. *The skew product flow $\{Z_t\}$ is all order mixing.*

Proof. To prove the theorem it suffices to show that the flow $\{Z_t\}$ is r-order mixing for any positive integer r. Let $f_0, \cdots, f_r$ be bounded measurable functions on $\Omega \times Y$ and $t_r^n, \cdots, t_r^n$ be real numbers satisfying the condition (3). We may restrict $f_j, 0 \leqq j \leqq r$, to the following functions:

$$\begin{aligned} &\varphi_0(\omega) = \exp\{-i \sum_{k=0}^{l_0} \theta_{0k}(x(a_{0k}, \omega) - x(a_{0k-1}, \omega))\} \\ (15) \qquad &\varphi_j(\omega) = \exp\{-i \sum_{k=1}^{l_j} \theta_{jk}(x(a_{jk}, \omega) - x(a_{jk-1}, \omega))\} \\ &\qquad a_{j0} < \cdots < a_{jP_j} = 0 < \cdots < a_{jl_j}, \\ &f_j(\omega, y) = \varphi_j(\omega)\psi_j(y), \end{aligned}$$

where θ_{jk}, $1 \leqq k \leqq l_j$, $0 \leqq j \leqq r$, are arbitrary real numbers and ψ_j, $0 \leqq j \leqq r$, are proper functions corresponding to proper values $\lambda_j, 0 \leqq j \leqq r$, of $\{S_t\}$. This is because the set of all finite linear combinations of such functions of the form (15) is dense in $L^2(\Omega)$ and the family $\mathfrak{A}$ of all proper functions of $\{S_t\}$ is dense in $L^2(Y)$. So the left hand side of (4) is equal to

$$\begin{aligned} &\left(\prod_{j=1}^{r} \psi_j, \bar{\psi}_0\right) \lim_{n\to\infty} \int \left\{ \prod_{j=1}^{r} \exp\left[i \sum_{k=1}^{l_j} \theta_{jk}(x(a_{jk} + t_j^n - t_0^n, \omega) \right.\right. \\ (16) \qquad &\left. - x(a_{jk-1} + t_j^n - t_0^n, \omega)) \right] \exp\left[i\lambda_j(x(t_j^n - t_0^n, \omega) - x(0, \omega))\right] \Big\} \\ &\cdot \exp\left[i \sum_{k=1}^{l_0} \theta_{0k}(x(a_{0k}, \omega) - x(a_{0k-1}, \omega)) \right] dm \end{aligned}$$

If $\lambda_0 = \cdots = \lambda_r = 0$, then the equality (4) is immediately obtained from the ergodicity of $\{S_t\}$ and the mixing property of $\{T_t\}$. Thus we consider the case that there are some non-zero proper values among the set $(\lambda_0, \cdots, \lambda_r)$. Take n large enough so that

$$\begin{aligned} a_{00} < \cdots < a_{0P_0} = 0 < \cdots < a_{0l_0} < a_{10} + t_1^n - t_0^n < \cdots < a_{1P_1-1} + t_1^n \\ - t_0^n < t_1^n - t_0^n < a_{1P_1+1} + t_1^n - t_0^n < \cdots < a_{1l_1} + t_1^n \\ - t_0^n < \cdots < a_{r0} + t_r^n - t_0^n < \cdots < a_{rP_r-1} + t_r^n \\ - t_0^n < t_r^n - t_0^n < a_{rP_r+1} + t_r^n - t_0^n < \cdots < a_{rl_r} + t_r^n - t_0^n. \end{aligned}$$

Then the integral in (16) is equal to

$$\begin{aligned} &\int \prod_{k=1}^{P_0} \exp\left[i\theta_{0k}(x(a_{0k}, \omega) - x(a_{0k-1} \cdot \omega))\right] dm \\ &\times \prod_{k=P_0+1}^{l_0} \int \exp\left[i(\theta_{0k} + \sum_{j=1}^{r} \lambda_j)(x(a_{0k}, \omega) - x(a_{0k-1}, \omega)) \right] dm \\ &\times \int \exp\left[i\left(\sum_{j=1}^{r} \lambda_j\right)(x(a_{10} + t_1^n - t_0^n, \omega) - x(a_{0l_0}, \omega)) \right] dm \end{aligned}$$

$$\times \prod_{m=1}^{r} \prod_{k=1}^{P_m} \int \exp\left[i\left(\theta_{mk} + \sum_{j=m}^{r} \lambda_j\right)(x(a_{mk} + t_m^n - t_0^n, \omega)\right.$$
$$\left. - x(a_{mk-1} + t_m^n - t_0^n, \omega))\right] dm$$
$$\times \prod_{m=1}^{r} \prod_{k=P_m+1}^{l_m} \int \exp\left[i\left(\theta_{mk} + \sum_{j=m+1}^{r} \lambda_j\right)(x(a_{mk} + t_m^n - t_0^n, \omega)\right.$$
$$\left. - x(a_{mk-1} + t_m^n - t_0^n, \omega))\right] dm$$
$$\times \prod_{k=2}^{r} \int \exp\left[i\left(\sum_{j=k}^{r} \lambda_j\right)(x(a_{k0} + t_k^n - t_0^n, \omega)\right.$$
$$\left. - x(a_{k-1, l_{k-1}} + t_{k-1}^n - t_0^n, \omega))\right] dm$$
$$= \prod_{k=1}^{P_0} \exp\left[-\tfrac{1}{2}\theta_{0k}^2(a_{0k} - a_{0k-1})\right]$$
$$\times \prod_{k=P_0+1}^{l_0} \exp\left[-\tfrac{1}{2}\left(\theta_{0k} + \sum_{j=1}^{r} \lambda_j\right)^2 (a_{0k} - a_{0k-1})\right]$$
$$\times \exp\left[-\tfrac{1}{2}\left(\sum_{j=1}^{r} \lambda_j\right)^2 (a_{10} + t_1^n - t_0^n - a_{0l_0})\right]$$
$$\times \prod_{k=2}^{r} \exp\left[-\tfrac{1}{2}\left(\sum_{j=k}^{r} \lambda_j\right)^2 (a_{k0} - a_{k-1, l_{k-1}} + t_k^n - t_{k-1}^m)\right]$$
$$\times \prod_{m=1}^{r} \prod_{k=1}^{P_m} \exp\left[-\tfrac{1}{2}\left(\theta_{mk} + \sum_{j=m}^{r} \lambda_j\right)^2 (a_{mk} - a_{mk-1})\right]$$
$$\times \prod_{m=1}^{r} \prod_{k=P_m+1}^{l_m} \exp\left[-\tfrac{1}{2}\left(\theta_{mk} + \sum_{j=m+1}^{r} \lambda_j\right)^2 (a_{mk} - a_{mk-1})\right] \to 0, n \to \infty,$$

because of the condition (3): $\lim_{n\to\infty} \min_{1\leq j\leq r} (t_j^n - t_{j-1}^n) = \infty$. Thus the conclusion follows.

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