251. *Some Existence Theorems in Cluster Set Theory*

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(Comm. by Kinjirô KUNUGI, M. J. A., Dec. 12, 1970)

1. Let C be the unit circle and D be the open unit disk in the complex plane.

Theorem 1. *There exists a holomorphic function f in D for which the set $I(f)$ of Plessner points* [3, p. 147] *is residual* [3, p. 75] *on C and of logarithmic measure* [7, p. 64] *zero.*

Theorem 2. *There exists a bounded univalent holomorphic function f in D for which the set $M(f)$ of Meier points* [3, p. 153] *is of logarithmic measure zero.*

Furthermore, we obtain some improvements of Bagemihl-Seidel's results [2, p. 191, Corollaries $3\sim 5$], one of which may be stated as

Theorem 3. *There exist a holomorphic function f in D and a subset S of C, being of logarithmic measure zero, such that the radial cluster set* [3, p. 72] *of f at any point of $C-S$ coincides with the unit circle.*

Remark 1. A bounded set of logarithmic measure zero is known to be of logarithmic capacity zero. In Remark 3 of the next section we ascertain this for our special example S.

I wish to express my warmest thanks to Prof. K. Hatano for valuable conversations.

2. We construct a subset S of C satisfying the following three conditions:

(i) $C-S$ is of first Baire category on C.

(ii) S is a G_δ subset of C.

(iii) The logarithmic measure of S is zero.

Let $K=\{z_1, \cdots, z_n, \cdots\}$ be a countable dense subset of C and let $\varepsilon_1, \cdots, \varepsilon_k, \cdots$ be a sequence of positive numbers such that $\varepsilon_k \to 0$ as $k\to\infty$. Let δ_{kn} be an open disk containing z_n whose radius is $r_{kn}=\exp(-2^n/\varepsilon_k)$ $(k, n=1, 2, \cdots)$. Let $\delta_k=\bigcup_{n=1}^{\infty}\delta_{kn}$ and let $\delta=\bigcap_{k=1}^{\infty}\delta_k$. Then $S=\delta\cap C$ is the desired one. Indeed, for any k, the closed set $C-\delta_k$ is nowhere dense on C since $\delta_k\cap C\supset K$ is open and dense on C. Therefore the set

$$(*)\qquad C-S=\bigcup_{k=1}^{\infty}(C-\delta_k)$$

is of first category on C. To prove (iii) we use the same notation as in [7, p. 63 ff.] with $h(t)=\{\log(1/t)\}^{-1}$. We use "disks" instead of "squares" in p. 63 of [7]; this is not an essential change. Then, since δ_{kn} ($n=1,2,\cdots$) cover S and $r_{kn}<\exp(-1/\varepsilon_k)\equiv\rho_k$, we have

$$H(\rho_k) \text{ (for } S) \leqq \sum_{n=1}^{\infty} h(r_{kn}) = \sum_{n=1}^{\infty} (\varepsilon_k/2^n) = \varepsilon_k.$$

This means that $H(\rho_k)\to 0$ as $k\to\infty$, which, combined with $\rho_k\to 0$ as $k\to\infty$, proves that the logarithmic measure of S is zero.

Remark 2. Generalizations to higher dimensional spaces or to metric spaces under the condition that C is a perfect set in the considered spaces are fairly easy. Prof. Hatano communicated orally that the condition (iii) may well be generalized to "Hausdorff measure".

Remark 3. By Erdös-Gillis's theorem [7, p. 66] combined with (iii) the inner logarithmic capacity and hence the logarithmic capacity of S is zero since S is capacitable by (ii). Consequently our example is simpler and more general than Kishi-Nakai's [4].

3. By Lappan's theorem [5, Theorem 2] there exists a holomorphic function f in D for which $I(f)=S$ since S is a G_δ subset of C. This proves Theorem 1.

We apply Bagemihl's theorem [1] to the set $C-S$. Then there exists a bounded univalent holomorphic function f in D such that $M(f)\cap(C-S)=\emptyset$ (empty). Theorem 2 follows from $M(f)\subset S$ and (iii).

Since $C-S$ is F_σ of first category and of the form of (*) on C we have the improvements of Corollaries $3\sim 5$ in [2, p. 191] in the sense that the subsets of C of exceptional character are of logarithmic measure zero.

Remark 4. Bagemihl's proof of his theorem in [1] depends in part on a theorem of Lohwater and Piranian [6, p. 7, Theorem 1′]. We remark that some theorems in [6] may well be applied to our F_σ set $C-S$. For example, we have:

There exists a bounded univalent holomorphic function f in D such that f has a radial limit $f(\zeta)$ at every point $\zeta\in C$ and that the function $f(\zeta)$ on C is discontinuous at every $\zeta\in C-S$ and is continuous at every $\zeta\in S$.

References

[1] F. Bagemihl: On the sharpness of Meier's analogue of Fatou's theorem. Israel J. Math., **4**, 230–232 (1966).

[2] F. Bagemihl and W. Seidel: Some boundary properties of analytic functions. Math. Z., **61**, 186–199 (1954).

[3] E. F. Collingwood and A. J. Lohwater: The Theory of Cluster Sets. Cambridge (1966).

[4] M. Kishi and M. Nakai: Linear set of the second category with zero capacity. Proc. Japan Acad., **44**, 633–635 (1968).

[5] P. Lappan: A characterization of Plessner points. Bull. London Math. Soc., **2**, 60–62 (1970).
[6] A. J. Lohwater and G. Piranian: The boundary behavior of functions analytic in a disk. Ann. Acad. Sci. Fenn. Ser. A I Math., **239**, 1–17 (1957).
[7] M. Tsuji: Potential Theory in Modern Function Theory. Tokyo (1959).